Maths for Science

This document, which was produced on 5th October 2003, is intended for evaluation purposes only. It is not a complete copy of *Maths for Science*. It contains Chapters 1 to 4 and the Glossary.

PPLATO PROMOTING PHYSICS LEARNING & TEACHING OPPORTUNITIES

The Open University

Contents

Introduction

Welcome to *Maths for Science*. There are many reasons for studying maths and a compelling motivation for many people is that it provides a way of representing and investigating the nature of the real world. Real world contexts could include population statistics, or economics, or engineering. Here, the context is 'science' in its broadest sense.

Much of science is couched in the language of mathematics. Nearly all courses in science will assume some mathematical skills and techniques. It is clearly not possible for *Maths for Science* to discuss all the mathematical techniques you might need to pursue your study of science to degree level, but by the end of it you will have acquired a good array of basic mathematical tools and confidence in using them. Equally importantly, you will have a foundation that should make it much easier to learn further mathematics if and when required.

Maths is in some sense a language with its own alphabet, vocabulary and 'rules of grammar'. With any language the only route to fluency is use and practice, but eventually the process of constructing or understanding sentences becomes automatic and one can then concentrate wholly on the message behind the words. You should aim to develop a similar confidence and fluency in carrying out certain important mathematical operations. There are few shortcuts: the route requires practice, practice and more practice! Keep paper, a pencil and your calculator to hand as you study, and use them constantly. You may find it helpful to write out notes and even to rework some of the examples given in the text as you go along. You will see that there are lots of questions seeded through the text and at the ends of sections; *you should work through each question as you reach it*. Links are provided to the solutions, but don't be tempted to look at these until you have made a serious attempt at working out the answer for yourself. If you have solved all parts of a question successfully on your own, then you are ready to move on.

The focus of *Maths for Science* is maths and not science, so you are not expected to bring specific prior knowledge of any particular branch of science. However, most of the examples and questions involve the application of mathematical tools to a real scientific purpose, so you will probably discover some interesting science along the way. Enjoy the journey!

1 Starting Points

The point to start from is always what you already know. It is assumed that you are familiar with the everyday usage of the basic arithmetic operations of addition, subtraction, multiplication, division and the use of a calculator to carry them out, decimal notation (e.g. for money), the representation of an idea by a formula (such as Einstein's famous $E = mc^2$), and the interpretation of information on a chart or graph (of the kind that might, for instance, accompany a TV news item about economic trends). Beyond that, you will find that many of the early chapters begin with a little revision of ideas and skills that you will probably already have met. This chapter, which concentrates on ideas about numbers – including fractions and powers – and the use of your calculator, is slightly different from later ones in that it covers concepts that are the basis for what is to follow in the rest of the course, so more of it may constitute revision.

If the points covered in the rest of this chapter are completely familiar, you need not spend very long on them, but they are worth checking out thoroughly as they are the foundation of much that is to come later in *Maths for Science*. Even if it is only for the sake of revision, make sure you understand all the emboldened terms and test your own skills against the learning outcomes by doing the numbered questions. If any of the material is new to you, time spent mastering it now will pay rich dividends later.

1.1 Numbers

'Numbers rule the universe' (Pythagoras)

Numbers are the bedrock of mathematics, underlying measurement, calculation and statistics, among other branches of maths. Everybody is familiar with the counting numbers (1, 2, 3, etc.), but scientists also make use of other kinds of numbers, so it is appropriate to begin this course with some revision of numbers of various sorts and the ways in which they may be combined.

1.1.1 Different types of number

One convenient way to represent numbers is on a ‘number line’, as shown in Figure 1.1. A ‘step’ to the right is taken by adding 1 to the previous number and a step to the left by subtracting 1. Positive and negative whole numbers, including zero, are called integers.

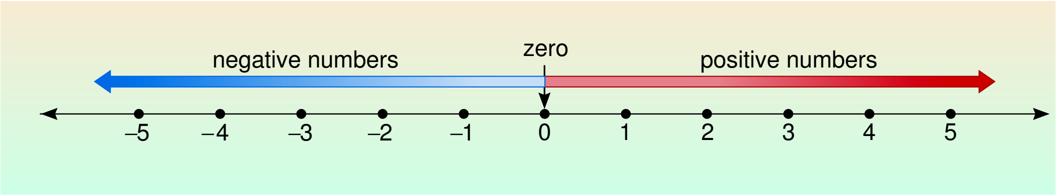


Figure 1.1: A number line showing the integers from −5 to 5.

Fractions (formed by dividing one integer by another) and their equivalent decimal numbers fit on the number line between the integers. For example, (i.e. 0.5) is halfway between 0 and 1, and −2.5 is halfway between −2 and −3. A number in which there is a decimal point (e.g. 0.5, 2.5, 100.35, etc.) is said to be written in decimal notation.

Figure 1.2 shows part of a thermometer. The inset portion covers a range from about +4.4 °C to −5.6 °C, which might represent the variation in temperature over a 24-hour period during the winter in the UK.

This illustrates how subdivision of the number line forms the basis of a scale for measuring physical quantities that can vary continuously. In this case, the scale between the integral values is divided into tenths. (Note that, in order to describe a physical quantity the numerical value has to be accompanied by a unit of measurement, in this case the degree Celsius. This aspect of measuring is covered in Chapters 2 and 3.)

In the case of a fraction such as $\frac{213}{25}$, the decimal equivalent is exact to two places of decimals (i.e. two digits after the decimal point):

$$\frac{213}{25} = 8.52$$

This decimal equivalent of $\frac{213}{25}$ cannot be given to more than two places of decimals except by putting zeros on the end (e.g. 8.520 000), so it is said to terminate at the digit 2.

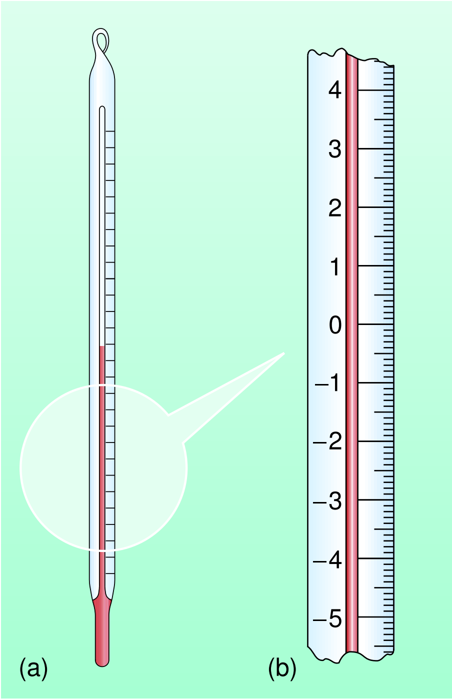
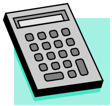


Figure 1.2: Part of a thermometer.

However, if you work out a fraction like $\frac{1}{3}$ on your calculator you will get a decimal like 0.333 333 333 (the number of digits displayed will depend on the make of your calculator). $\frac{41}{333}$ will come out as 0.123 123 123, and $\frac{70}{9}$ as 7.777 777 778. These decimals in fact recur (i.e. repeat themselves) for ever, so they are called infinite recurring decimals. The calculator truncates them when it runs out of digits on the display, and in the case of the final example also 'rounds up' the last digit from a 7 to an 8. In scientific calculations, it is usually totally inappropriate to quote so many digits after the decimal point and in Chapter 2 we will consider the rules for deciding how to round off such numbers in real situations.

Fractions and decimals are grouped together as the so-called rational numbers. All the rational numbers result in a decimal that either terminates or recurs. However, there are also numbers whose decimal equivalent neither terminates nor recurs. These numbers cannot be obtained by dividing one integer by another, so they are called irrational numbers. Well-known examples are $\sqrt{2}$ (the number that multiplied by itself gives 2, said as 'the square root of 2') and (π, which is defined as the number obtained by dividing the circumference of a circle by its diameter). This would be an appropriate moment to check that you know how to use the π button on your calculator. You should be able to get:

$$2 \times \pi = 6.283\,185\,307$$



Note that as there are so many makes of scientific and graphics calculators on the market, each operating differently, it is impossible to state the exact sequence of keystrokes you will need to carry out particular calculations. Whenever you meet a new type of mathematical operation, you should therefore check that you know how to perform it on your own calculator and refer to the manufacturer's instruction book if necessary. A calculator symbol in the margin will alert you to the points at which you particularly need to carry out this kind of check.

All the integers, rational and irrational numbers can be placed somewhere on the number line, so they are grouped together as the real numbers. All the numbers you will use in this course will be real. However, it may interest you to know that there are also imaginary numbers based on the square root of minus 1, which is usually represented by the symbol i. Numbers made up of real and imaginary parts, such as $(3 + 2i)$ are known as complex numbers. Complex numbers are used quite extensively in science and have practical applications in telecommunications, electrical engineering and the beautiful patterns of fractals.

In case hearing about all these different types of numbers leads you to think that straightforward 'counting numbers' hold little interest for scientists, Box 1.1 shows how a series of numbers, which mathematicians find interesting in their own right, have also been found to describe intricate patterns of plant growth.

Box 1.1 Fibonacci numbers

The sequence of numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 \ldots$$

was first defined in 1202 by the Italian mathematician Leonardo of Pisa, nicknamed Fibonacci. Each term in the sequence after the first two is obtained by adding together the previous two ($0+1=1$; $1+1=2$; $1+2=3$; $2+3=5$, etc.)

It is intriguing to discover that the spiral patterns of plant growth correspond to pairs of numbers in this series, as illustrated in Figure 1.3.

Part (a) shows a pinecone with 8 parallel rows of bracts spiralling gradually and 13 parallel rows of bracts spiralling steeply.

Part (b) shows a sunflower head in which the seeds spiral out from the centre: 55 rows clockwise and 89 rows anticlockwise.

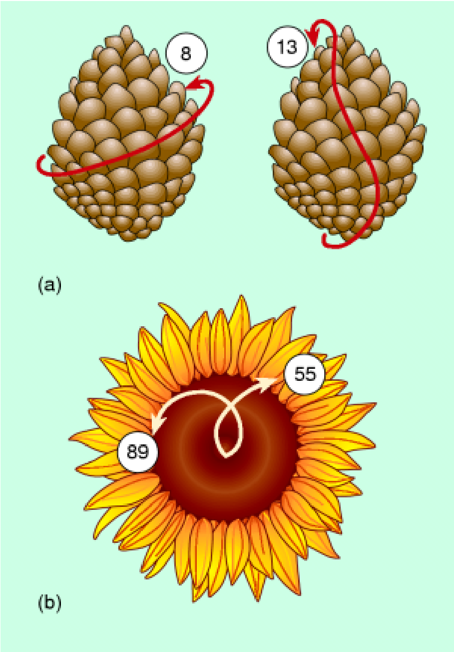


Figure 1.3: Fibonacci numbers in nature.

1.1.2 Calculating with negative numbers

Many scientific quantities can take negative values. For example, chemical reactions may either give out heat to the surroundings or absorb heat from the surroundings. Scientists adopt a convention that in the case of a heat-absorbing reaction, the change in energy has a positive value. In the case of a heat-releasing reaction (such as combustion), on the other hand, the energy change is negative. To be able to handle quantities like this in scientific calculations it is essential to understand the rules for performing the arithmetic operations (addition, subtraction, multiplication and division) when negative numbers are involved. If I amalgamate a credit card debt of £100 with an overdraft of £150, I owe £250 in total:

$$\pounds 100\,\text{debt} + \pounds 150\,\text{debt} = \pounds 250\,\text{debt}$$

Just in terms of numbers, this is equivalent to writing:

$$(-100) + (-150) = -250$$

Note from this example how brackets can be used to make it clear how numbers and signs are associated. The rules for performing arithmetic operations with negative numbers are summarized by the examples in the box 'Arithmetic with negative numbers'. You should check that you are familiar with all the rules exemplified in the box.

Arithmetic with negative numbers

The numbers used as examples here are small integers between 1 and 10, but could of course be any number. As is normally the case, positive numbers are not preceded by a + sign.

$$
\begin{array}{lll}
(-3) + 5 = 2 & 3 + (-4) = 1 & (-3) + (-3) = -6 \\
(-5) - 2 = 7 & 4 - (-3) = 7 & (-5) - (-4) = -1 \\
(-2) \times 2 = -4 & 3 \times (-2) = 6 & (-2) \times (-2) = 4 \\
(-3) \div 3 = -1 & 3 \div (-3) = -1 & (-3) \div (-3) = 1
\end{array}
$$

Thinking about some of the examples in concrete terms may help to make sense of them. For instance, taking money from a bank account that is already overdrawn increases the amount of the debt (i.e. makes it ‘more negative’). Doubling an overdraft produces an even larger debt (i.e. a bigger negative number).

Brackets are included to associate negative signs with particular numbers. For example, $3+(-4)$ means that (-4) is being added to 3; this is equivalent to subtracting 4 from 3, with the result (1).

Before reading on, test your understanding of the rules by doing Question 1.1.

Question 1.1

Without using your calculator, work out:

(a) $(-3) \times 4$ Answer

(b) $(-10) - (-5)$ Answer

(c) $6 \div (-2)$ Answer

(d) $(-12) \div (-6)$ Answer

The examples given so far illustrate one important feature of both addition and multiplication: both these operations are commutative. This is just the mathematical way of saying that if one adds two numbers then the result (called the sum) is identical whichever number is written first. For example:

$$5 + 3 = 8 \text{ and } 3 + 5 = 8$$
$$(-2) + 3 = 1 \text{ and } 3 + (-2) = 1$$

Similarly, in multiplying two numbers the result (called the product) is unchanged if the order of the numbers is reversed. For instance:

$$5 \times 4 = 20 \text{ and } 4 \times 5 = 20$$
$$(-3) \times 4 = -12 \text{ and } 4 \times (-3) = -12$$

Subtraction and division, on the other hand, are not commutative:

$$5 - 3 = 2 \text{ but } 3 - 5 = -2$$
$$8 \div 4 = 2 \text{ but } 4 \div 8 = \tfrac{1}{2}$$

The commutativity of addition and multiplication may seem rather obvious when applied to the counting numbers, but is worthy of attention because of its importance in the algebraic manipulations that will be discussed in Chapter 4.

Worked example 1.1 and Question 1.2 are two rather more realistic examples requiring the use of arithmetic with negative numbers.

Worked example 1.1

One of the hottest places on Earth is Death Valley, California, where an air temperature of 56 °C has been recorded. Probably the coldest inhabited place is the Siberian village of Oymyakon, where the temperature has fallen to −72 °C. What is the difference in temperature between these two extremes?

Answer

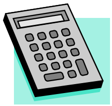
The difference in temperature may be worked out in two ways. The first method involves subtracting the lower temperature from the higher, i.e. 56 °C − (−72 °C), which gives a *positive* difference of 128 Celsius degrees. This is the amount by which Death Valley is hotter than Oymyakon. Alternatively, it is equally valid to subtract the higher temperature from the lower, i.e. −72 °C − 56 °C, which gives a *negative* difference of −128 Celsius degrees. This is equivalent to saying that Oymyakon is 128 Celsius degrees colder than Death Valley.

This example shows that in scientific calculations involving negative numbers it is important to keep the physical situation in mind.

Question 1.2 Answer

The maximum temperature range within the oceans is 31.9 Celsius degrees. This is a much smaller variation in temperature than that achievable for the air above a landmass, in part because the lowest ocean temperature is fixed at the temperature at which seawater freezes. The highest recorded ocean temperature is 30.0 °C. What is the freezing point of seawater?

1.1.3 Working with negative numbers on a calculator



The calculations in Questions 1.1 and 1.2 were easy enough to work out by hand, but many of the calculations you will encounter in science will require the use of a calculator. It is therefore important to check that you know how to input negative numbers into your own calculator.

Take the following examples:

$6 + (-8) = -2$

$4 - (-3) = 7$

$5 \times (-3) = -15$

$(-8) \div (-2) = 4$

and make sure that you can carry out each sum on your calculator, obtaining the correct sign on the display of the answer. With some makes of calculator you will be able to enter the expression on the left-hand side more or less as it is written, with or without brackets. With other makes you may have to use a combination of the arithmetic operation keys and the $+/-$ (or on some makes $\pm$) button.

When you are confident that you can input negative numbers in association with the first arithmetic operations, test your skill with Question 1.3.

Question 1.3

Making sure you input all the signs, use your calculator to work out the following:

(a) $117 - (-38) + (-286)$ Answer

(b) $(-1624) \div (-29)$ Answer

(c) $(-123) \times (-24)$ Answer

There is, however, one case in which the calculator does not fully deal with signs, and that case concerns square roots. The 'square root of 9' is defined as the number that multiplied by itself gives 9. One such number is 3:

$$3 \times 3 = 9$$

and if you use your calculator to work out $\sqrt{9}$ you will indeed obtain the answer 3. However, it is also true that

$$-3 \times -3 = 9$$

So the square root of 9 is either +3 or −3. It is a mathematical convention that the notation $\sqrt{9}$ means 'the positive value of the square root of 9', and this is what your calculator displays. In cases in which the negative value of the square root might be relevant this is indicated by use of the sign ± (plus or minus) before the square root sign, i.e. $\pm\sqrt{9}$.

In Section 1.1.1, the number $\sqrt{2}$ was given as an example of an irrational number. Check that you can use the square root button on your own calculator to get

$$\sqrt{2} = 1.414\,213\,562$$

(You may obtain more or fewer digits depending on the make and model of your calculator. The fact that the number is irrational means that in any case it never ends.)

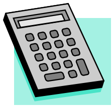
Question

What is $\dfrac{\sqrt{5}}{3}$?

Answer

$$\frac{\sqrt{5}}{3} = 0.745\,355\,922$$

Be sure to check that you can obtain this value on your own calculator, by ensuring that the calculator takes the square root of 5 *before* dividing by 3. Otherwise, you



will get the positive value of the square root of $\frac{5}{3}$, which is not the same at all!

$$\sqrt{\frac{5}{3}} = 1.290\,994\,449$$

1.1.4 The number zero

Zero is a number to be careful about, especially when it is used in multiplication or division.

If you try multiplying 0 by 6 on your calculator, you will get the answer 0. This is hardly surprising. If we start off with nothing, it doesn't matter how often we multiply it, we still have nothing. The commutativity of multiplication shows that 6×0 is therefore also equal to 0, and your calculator will confirm this.

> The result of multiplying any number by 0 is 0.

In a similar way, dividing 0 by any non-zero number gives zero.

Trying to divide by zero is more problematic. If you enter $6 \div 0$ into your calculator, you will get an error message. To understand why, imagine dividing 6 by successively smaller and smaller numbers: the answers will get successively larger and larger. The number by which we're dividing approaches zero, the result of the division becomes too large for the calculator to cope with. Dividing by zero does not produce a meaningful number and is to be avoided!

1.2 Fractions

With the increasing decimalization of everyday units of measurement, we use fractions less than people used to. Nowadays adding eighths and sixteenths of inches is about as much as you might need to do, and that only if you still have a ruler, or some items in a toolbox, marked in inches. However the ability to add, subtract, multiply and divide using numerical fractions is extremely important in *Maths for Science*, because it is the basis for the skill of manipulating *algebraic* fractions which will be discussed in Chapter 4.

1.2.1 Using fractions

Fractions are characterized by a numerator (the number on top) and a denominator (the number on the bottom). So in the fraction $\frac{3}{8}$, the numerator is 3 and the denominator is 8.

A pictorial representation, such as that in Figure 1.4, makes it obvious that it is possible to have fractions which have different numerators and denominators, but are nevertheless equal. The cake can be divided into two and the shaded half further sub-divided into two quarters or four eighths, but half the cake still remains shaded. So the fractions $\frac{1}{2}$, $\frac{2}{4}$ and $\frac{4}{8}$ all represent the same amount of the original cake, and can therefore be described as equivalent fractions.

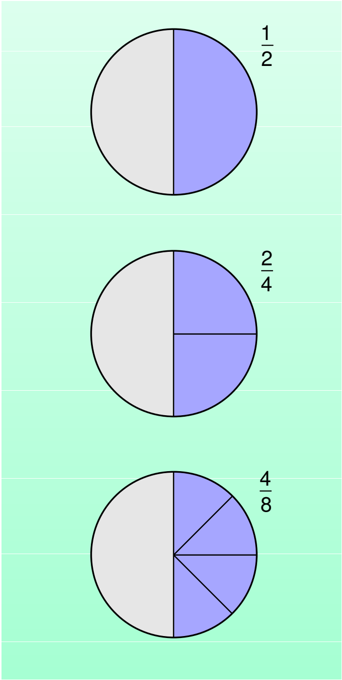


Figure 1.4: Sharing out half a cake.

Figure 1.4 exemplifies the most fundamental rule associated with fractions:

> The value of a fraction is unchanged if its numerator and denominator are both multiplied by the same number, or both divided by the same number.

In the case of the half cake, numerator and denominator have been multiplied by 2 to get the equivalent two quarters and again to get the equivalent four eighths. In the following example of equivalent fractions, other multiplying and dividing numbers have been used:

$$\frac{6}{9} = \frac{2}{3} = \frac{8}{12} = \frac{10}{15}$$

$\frac{2}{3}$ is the simplest form in which this fraction may be expressed, i.e. the one in which the numerator and denominator have the smallest possible value.

A percentage means a 'number of parts per hundred', so is equivalent to a fraction in which the denominator is 100. For example, 50% is the same as $\frac{50}{100}$ or $\frac{1}{2}$

Question

Express 35% as a fraction of the simplest possible form.

Answer

35% is the same as $\frac{35}{100}$. The value of the faction will be unchanged if the numerator and denominator are both divided by the same number, and 35 and 100 can both be divided by 5. Doing this gives

$$\frac{35}{100} = \frac{7}{20}$$

This is the simplest form in which the fraction can be expressed.

One way to convert a fraction to a percentage is to multiply top and bottom of the fraction by whatever number is required to make the denominator equal to 100. For instance:

$$\frac{1}{4} = \frac{1 \times 25}{4 \times 25} = \frac{25}{100}$$

Hence $\frac{1}{4}$ is equivalent to 25%.

In the first few sections of this course, all fractions have been written in the form $\frac{3}{4}$. However, in most maths and science texts, you will find that the alternative form, 3/4, is also very common, so you have to become equally comfortable with both systems and also have to be able to swap between them at will. From now on, therefore, both notations will be used.

1.2.2 Adding and subtracting fractions

Suppose we want to add the two fractions shown below:

$$\frac{3}{4} + \frac{7}{16}$$

We cannot just add the 3 and the 7. The 3 represents 3 'quarters' and the 7 represents 7 'sixteenths', so adding the 3 to the 7 would be like trying to add 3 apples and 7 penguins!

In order to add or subtract two fractions, it is necessary for them both to have the same *denominator* (bottom line).

Fractions with the same denominator are said to have a common denominator. In numerical work, it is usually convenient to pick the smallest possible number for this denominator (the so-called lowest common denominator). In this example, the lowest common denominator is 16; we can multiply both top and bottom of the fraction $\frac{3}{4}$ by 4 to obtain the equivalent fraction $\frac{12}{16}$, so the calculation becomes

$$\frac{3}{4} + \frac{7}{16} = \frac{12}{16} + \frac{7}{16} = \frac{19}{16}$$

A top heavy fraction such $\frac{19}{16}$ (i.e. one in which the numerator is larger than the denominator) is sometimes referred to as an improper fraction. We could also write the final answer as $1\frac{3}{16}$. This notation is called a mixed number (i.e. a combination of a whole number and a simple fraction). However for most purposes in this course it is better to leave things as improper fractions.

If the lowest common denominator is not easy to spot, it is perfectly acceptable to use *any* common denominator when adding and subtracting fractions. It may be most convenient to multiply the top and bottom of the first fraction by the denominator of the second fraction, and the top and bottom of the second fraction by the denominator of the first. A return to our example may make this clearer:

$$\frac{3}{4}+\frac{7}{16}=\frac{3\times 16}{4\times 16}+\frac{7\times 4}{16\times 4}=\frac{48}{64}+\frac{28}{64}=\frac{76}{64}$$

However, $\frac{76}{64}$ is not the simplest form in which this fraction can be expressed. We can divide both the numerator and the denominator by four to obtain $\frac{19}{16}$. Reassuringly, this is the same answer as we obtained before!

This process of dividing the top and bottom of a fraction by the same quantity is often referred to as cancellation, because it is commonly shown by striking through the numbers being divided. For example, $\frac{5}{15}$ can be simplified by dividing the numerator and denominator by 3, and this may be shown as

$$\frac{\not{5}\,^{1}}{\not{1}5\,_{3}}$$

Worked example 1.2

Evaluate $\frac{3}{2}+\frac{1}{32}$, giving the answer in the form of the simplest possible improper fraction.

Note that the instruction to 'evaluate' simply means 'calculate the value of'.

Answer

Choosing 2×32 as the common denominator,

$$\begin{aligned}\frac{3}{2}+\frac{1}{32} &= \frac{3\times 32}{2\times 32}+\frac{1\times 2}{32\times 2}\\ &= \frac{96}{64}+\frac{2}{64}\\ &= \frac{98}{64}\\ &= \frac{\cancel{98}^{49}}{\cancel{64}_{32}}\end{aligned}$$

This cannot be simplified any further, so

$$\frac{3}{2}+\frac{1}{32}=\frac{49}{32}$$

Question 1.4

Without using a calculator, evaluate the following, leaving your answers in the form of the simplest possible fractions.

(a) $\dfrac{2}{3} - \dfrac{1}{6}$ Answer

(b) $\dfrac{1}{3} + \dfrac{1}{2} - \dfrac{2}{5}$ Answer

(c) $\dfrac{5}{28} - \dfrac{1}{3}$ Answer

1.2.3 Manipulating fractions

It is very important to remember that multiplying both numerator and denominator by the same non-zero number, or dividing both numerator and denominator by the same non-zero number, are the *only* things you can do to a fraction that leave its value unchanged. Adding the same number to the numerator and denominator will alter the value of the fraction, as will any other operations. The following question will help you to convince yourself of this, so it is particularly important that you should work through it at this point.

Question 1.5

Take any fraction, say $\frac{4}{16}$, and evaluate it as a decimal, using your calculator if necessary. Now try each of the following operations in turn, using your calculator to work out the result:

(a) choose any integer and add it to the numerator and denominator Answer

(b) subtract the same integer from the numerator and denominator Answer

(c) square the numerator and the denominator (i.e. multiply the numerator by itself, and the denominator by itself) Answer

(d) take the square root of the numerator and the square root of the denominator. Answer

The results you obtained for Question 1.5 confirm that, for example, adding the same non-zero number to the top and bottom of a fraction changes its value, as do operations such as taking the square root of the numerator and denominator. The experience of all calculations of this type can be generalized by saying that *excluding operations involving the integer zero,*

A fraction is unchanged by either the multiplication, or the division, of its numerator and denominator by the same amount. All other operations carried out on the fraction will alter its value.

In terms of numerical fractions, this rule may seem fairly obvious. But forgetting it once the numbers are replaced by symbols is the root cause of many errors in algebra!

1.2.4 Multiplying fractions

The expression ‘three times two’ just means there are three lots of two (i.e. $2+2+2$). So multiplying by a whole number is just a form of repeated addition. For example,

$$3 \times 2 = 2 + 2 + 2$$

This is equally true if you are multiplying a fraction by a whole number:

$$3 \times \frac{4}{5} = \frac{4}{5} + \frac{4}{5} + \frac{4}{5} = \frac{12}{5}$$

We could write the 3 in the form of its equivalent fraction $\frac{3}{1}$ and it is then clear that the same answer is obtained by multiplying the two numerators together and the two denominators together.

$$\frac{3}{1} \times \frac{4}{5} = \frac{3 \times 4}{1 \times 5} = \frac{12}{5}$$

In fact, this procedure holds good for any two fractions.

To multiply two or more fractions, multiply the numerators (top lines) together and also multiply the denominators (bottom lines) together.

So

$$\frac{3}{4} \times \frac{7}{8} = \frac{3 \times 7}{4 \times 8} = \frac{21}{32}$$

Multiplying three fractions together is done by simple extension of the method used in the previous examples:

$$\frac{7}{16} \times \frac{7}{8} \times \frac{3}{4} = \frac{7 \times 7 \times 3}{16 \times 8 \times 4} = \frac{147}{512}$$

1.2.5 Dividing fractions

How are we to interpret $4 \div \frac{1}{2}$? The analogy with dividing by an integer may help. The expression $4 \div 2$ asks us to work out how may twos there are in 4 (answer 2). In exactly the same way, the expression $4 \div \frac{1}{2}$ asks how many halves there are in 4. Figure 1.5 illustrates this in terms of circles. Each circle contains two half-circles, and 4 circles therefore contain 8 half-circles. So

$$4 \div \frac{1}{2} = 4 \times 2 = 8$$

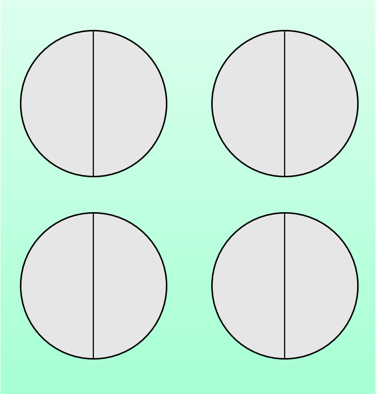


Figure 1.5: Each circle contains two half-circles.

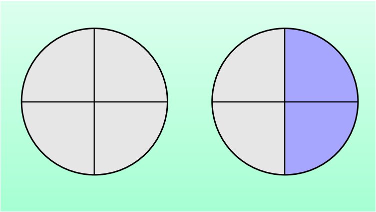


Figure 1.6: Each half-circle contains two quarter-circles.

Similarly, $\frac{1}{2} \div \frac{1}{4}$ asks how many quarters there are in a half. Figure 1.6 illustrates that:

- each whole circle contains 4 quarter-circles
- each half-circle contains $\frac{1}{2} \times 4$ quarter-circles

So

$$\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \times 4 = \frac{1}{2} \times \frac{4}{1} = \frac{1 \times 4}{2 \times 1} = \frac{4}{2} = 2$$

This may be extended into a general rule

To divide by a fraction, turn it upside down and multiply.

So

$$\frac{4}{3} \div \frac{5}{9} = \frac{4}{3} \times \frac{9}{5} = \frac{\cancel{36}^{12}}{\cancel{15}_{5}} = \frac{12}{5}$$

Here the cancellation has been done by dividing the numerator and the denominator of the final answer by 3. However, cancellation could equally well have been carried out at an earlier stage,

$$\frac{4}{\cancel{3}_1} \times \frac{\cancel{9}^3}{5} = \frac{12}{5}$$

Note that divisions involving fractions are commonly written in several different ways; the example above might equally well have been expressed as $\frac{4}{3}\Big/\frac{5}{9}$ or $\frac{4/3}{5/9}$.

It is always important to remember that an integer is equivalent to a fraction in which the numerator is equal to that integer and the denominator is equal to 1: for example, the integer 3 is equivalent to the fraction $\frac{3}{1}$. So dividing by the integer 3 is equivalent to dividing by the fraction $\frac{3}{1}$, and that, according to the general rule about how to divide by a fraction, is the same as multiplying by the fraction $\frac{1}{3}$.

Thus $\dfrac{1}{2} \div 3 = \dfrac{1}{2} \div \dfrac{3}{1} = \dfrac{1}{2} \times \dfrac{1}{3} = \dfrac{1 \times 1}{2 \times 3} = \dfrac{1}{6}$

In this context, it may be helpful to restate the general rule in terms of a specific example:

Multiplying by $\frac{1}{2}$ is equivalent to dividing by 2.

Dividing by $\frac{1}{2}$ is equivalent to multiplying by 2.

The blue box and the cartoon use the integer 2 as the example, but it could of course be replaced by any other integer: it is equally true to say that dividing by $\frac{1}{10}$ is equivalent to multiplying by 10.

Question 1.6

Work out each of the following, leaving your answer as the simplest possible fraction:

(a) $\frac{2}{7} \times 3$ Answer

(b) $\frac{5}{9} \div 7$ Answer

(c) $\frac{1/6}{1/3}$ Answer

(d) $\frac{3}{4} \times \frac{7}{8} \times \frac{2}{7}$ Answer

1.3 Powers, reciprocals and roots

1.3.1 Powers

Most people are familiar with the fact that 2×2 can also be written as 2^2 (said as 'two squared') and $2 \times 2 \times 2$ as 2^3 (said as 'two cubed'). This shorthand notation can be extended indefinitely, so $2 \times 2 \times 2 \times 2 \times 2 \times 2$ becomes 2^6 (said as 'two raised to the power of six' or 'two to the power of six', or more usually just as 'two to the six'). In these examples, 2 is called the base number and the superscript indicates the number of '2's that have been multiplied together. The superscript number is variously called the exponent, the index (plural indices) or the power. In the rest of this section, the term exponent will be the one used, because that ties in most closely with the notation on calculators.

'Power' is a slightly confusing term because it is commonly used to denote two different quantities:

- the value of the superscript number (as in 'two to the power of six'),
- the complete package of base number and exponent .

The context should make it clear what is meant in any particular example.

In the following example, the base number is 5:

Exponent	1	2	3	4
Power of 5	5^1	5^2	5^3	5^4
Value	5	25	125	625

If you read this table starting at the right and stepping to the left, each time you take a step you are subtracting 1 from the number in the top row and dividing the number in the bottom row by five. On the basis of this pattern, mathematicians extend this table further to the left by continuing to apply the same 'rule' for each step, giving:

Exponent	−3	−2	−1	0	1	2	3	4
Power of 5	5^{-3}	5^{-2}	5^{-1}	5^0	5^1	5^2	5^3	5^4
Value	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25	125	625

Firstly, note the extremely important result that $5^0 = 1$.

Any base number raised to the power of zero is equal to 1.

Next, notice that $5^{-2} = \frac{1}{25}$. But since $25 = 5^2$, $\frac{1}{25}$ is also $\dfrac{1}{5^2}$. So we have developed a new form of shorthand such that

$$5^{-1} = \frac{1}{5} \qquad 5^{-2} = \frac{1}{5^2} \qquad 5^{-3} = \frac{1}{5^3} \qquad \text{and so on.}$$

Another way of saying this is that 5^{-2} is the reciprocal of 5^2. The reciprocal of any number is 1 divided by that number. Note that this also works the other way round: 5^2 is the reciprocal of 5^{-2}. In other words $5^2 = \dfrac{1}{5^{-2}}$.

The system shown above for powers of 5 could be applied to any base number, and is especially useful when applied to powers of ten, because then it ties in with our normal system for writing decimal numbers. In the example below, the table is constructed the other way round to emphasise this:

	thousands	hundreds	tens	units	point	tenths	hundredths	thousandths
Value	1000	100	10	1	.	0.1	0.01	0.001
Power of 10	10^3	10^2	10^1	10^0		10^{-1}	10^{-2}	10^{-3}
Exponent	3	2	1	0		−1	−2	−3

In the next chapter, you will see how useful this powers of ten notation can be in scientific work.

Question 1.7

Without using a calculator, evaluate

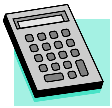
(a) 2^{-2} Answer

(b) $\dfrac{1}{3^{-3}}$ Answer

(c) $\dfrac{1}{4^0}$ Answer

(d) $\dfrac{1}{10^4}$ Answer

Your calculator probably has an x^2 button, and either an x^{-1} or a $1/x$ button, but to evaluate other powers you will have to use a special 'powers' button. On some calculators this is marked x^y, on others it has the symbol $\wedge$. To input a negative exponent, you may have to combine the powers button with the $+/-$ button. Make sure at this point that you can operate your own calculator to obtain correctly:



$$5^4 = 625$$

$$5^{-1} = 0.2 \text{ (i.e. } 1/5)$$

$$5^{-2} = 0.04 \text{ (i.e. } 1/25)$$

Question 1.8

Use your calculator to evaluate:

(a) 2^9 Answer

(b) 3^{-3} Answer

(c) $\frac{1}{4^2}$ Answer

Box 1.2 An intimate knowledge of powers!

Srinivasa Ramanujan (1887–1920), an Indian mathematician of immense talent, came to England in 1913 at the invitation of the distinguished British mathematician, G. H. Hardy. In his biography of Ramanujan, Hardy wrote:

I remember once going to see him when he was lying ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."

Indeed: $1729 = 1^3 + 12^3 = 9^3 + 10^3$

1.3.2 Multiplying and dividing with powers

In scientific calculations, it is very common to have to multiply and divide by powers, especially powers of ten. It is therefore extremely important to become confident in manipulating powers in this way, both with and without a calculator. However, the rules for doing so are quite easy to work out.

Suppose we wanted to multiply 10^3 by 10^2. We could write this out more fully as

$$10^3 \times 10^2 = (10 \times 10 \times 10) \times (10 \times 10) = 10^5$$

The exponent of the result (5) is the same as the sum of the two original exponents $(3 + 2)$.

The process is of course not limited to powers of ten. It works for any base number. For example:

$$2^2 \times 2^4 = (2 \times 2) \times (2 \times 2 \times 2 \times 2) = 2^6$$

Again, the exponent of the result (6) is the same as the sum of the two original exponents $(2 + 4)$.

The process also works for negative exponents. For example, since $5^{-2} = \dfrac{1}{5^2}$

$$5^3 \times 5^{-2} = (5 \times 5 \times 5) \times \frac{1}{5 \times 5} = 5 = 5^1$$

Adding the exponents here again gives the exponent of the answer:

$$3 + (-2) = 1$$

In science and maths, general rules are often stated in terms of symbols. We could express the rule we have discovered through the above examples in the much more general form

$$N^a \times N^b = N^{a+b} \tag{1.1}$$

where N represents any base number and a and b represent any exponents

Quantities such as those represented by the symbols N, a and b, which can take any value we choose, are called variables.

The example involving a negative exponent we looked at previously shows immediately how to extend the rules to cover situations in which we want to divide powers. We had:

$$5^3 \times 5^{-2} = 5^{3+(-2)} = 5^1 = 5$$

But as you will remember from Section 1.2.5, multiplying by a fraction is the same as dividing by that fraction turned upside down (i.e. its reciprocal). So multiplying by 5^{-2} is the same as dividing by its reciprocal (5^2), and we can write

$$5^3 \div 5^2 = 5^{3-2} = 5^1 = 5$$

This time, instead of adding the exponents, we have subtracted the second from the first. More generally,

$$N^a \div N^b = N^{a-b} \qquad (1.2)$$

where N represents any base number and a and b represent any exponents

Question 1.9

Without using a calculator, simplify the following to the greatest possible extent (leaving your answer expressed as a power).

(a) $2^{30} \times 2^2$ Answer

(b) $3^{25} \times 3^{-9}$ Answer

(c) $10^2/10^3$ Answer

(d) $10^2/10^{-3}$ Answer

(e) $10^{-4} \div 10^2$ Answer

(f) $\dfrac{10^5 \times 10^{-2}}{10^3}$ Answer

1.3.3 Powers of powers

Consider now what happens when a number which is already raised to a power, for example 3^2, is again raised to a power. Suppose for example 3^2 is itself cubed, so that we have $\left(3^2\right)^3$. Writing this out in full shows that

$$\left(3^2\right)^3 = (3^2) \times (3^2) \times (3^2) = (3 \times 3) \times (3 \times 3) \times (3 \times 3) = 3^6$$

This time the exponents have been multiplied together to obtain the exponent of the answer: $3 \times 2 = 6$.

More generally,

$$(N^m)^n = N^{m \times n} \tag{1.3}$$

where N represents any base number and m and n represent any exponents

Equation 1.3 applies for all values of N, m and n whether positive or negative. So for example:

$$\left(\frac{1}{10^{20}}\right)^3 = \left(10^{-20}\right)^3 = 10^{(-20)\times 3} = 10^{-60} = \frac{1}{10^{60}}$$

This is equivalent to saying that

$$\left(\frac{1}{10^{20}}\right)^3 = \frac{1^3}{(10^{20})^3} = \frac{1}{10^{20\times 3}} = \frac{1}{10^{60}}$$

Question 1.10

Without using a calculator, simplify the following to the greatest possible extent, leaving your answer expressed as a power.

(a) $\left(4^{16}\right)^2$ Answer

(b) $\left(5^{-3}\right)^2$ Answer

(c) $\left(10^{25}\right)^{-1}$ Answer

(d) $\left(\frac{1}{3^3}\right)^6$ Answer

1.3.4 Roots and fractional exponents

Finally, how are we to interpret a power with a fractional exponent, such as $2^{1/2}$? The rule for multiplying powers gives a clue. Suppose we were to multiply $2^{1/2}$ by itself. Applying Equation 1.1 suggests that:

$$2^{1/2} \times 2^{1/2} = 2^{\left(\frac{1}{2}+\frac{1}{2}\right)} = 2^1 = 2$$

But the positive number that multiplied by itself gives 2 is more commonly written as $\sqrt{2}$. The two shorthands, $2^{1/2}$ and $\sqrt{2}$ are often used interchangeably.

Similarly, the number that multiplied by itself three times gives 125 is sometimes written as $\sqrt[3]{125}$ (said as 'the cube root of 125'), but more commonly written in science as $(125)^{1/3}$. This number is clearly 5, and you should notice the correspondence:

$$5^3 = 125 \text{ and conversely } (125)^{1/3} = 5$$

More generally,

The positive nth root of a number N can be written as either $\sqrt[n]{N}$ or as $N^{1/n}$

In practice, the first type of notation is only used when $n = 2$ or $n = 3$.

Worked example 1.3

Without using a calculator, evaluate $\dfrac{\left(2^{1/2}\right)^7}{(2^3)^{1/2}}$

Answer

From Equation 1.3

$$\left(2^{1/2}\right)^7 = 2^{\frac{1}{2}\times 7} = 2^{7/2} \quad \text{and} \quad \left(2^3\right)^{1/2} = 2^{3\times\frac{1}{2}} = 2^{3/2}$$

so

$$\frac{\left(2^{1/2}\right)^7}{(2^3)^{1/2}} = \frac{2^{7/2}}{2^{3/2}}$$

From Equation 1.2

$$\begin{aligned}\frac{2^{7/2}}{2^{3/2}} &= 2^{7/2} - 2^{3/2} \\ &= 2^{4/2} \\ &= 2^2 \\ &= 4\end{aligned}$$

Equation 1.3 can now be used to bring meaning to a number like $27^{2/3}$.

Since $\frac{2}{3} = \frac{1}{3} \times 2$, applying Equation 1.3 shows that $27^{2/3} = (27^{1/3})^2$ i.e. the square of the cube root of 27. The cube root of 27 is 3, so $27^{2/3}$ is equal to 3^2 or 9.

Question 1.11

Without using a calculator, simplify the following to the greatest possible extent, expressing your answer as an integer or a decimal.

(a) $\left(2^4\right)^{1/2}$ Answer

(b) $\sqrt{10^4}$ Answer

(c) $100^{3/2}$ Answer

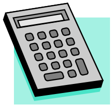
(d) $(125)^{-1/3}$ Answer

1.4 Doing calculations in the right order

In Section 1.1.2, brackets were used to make it clear that the minus signs were tied to particular numbers. Brackets can also be used to show the order in which calculations are to be performed.

If a calculation were written as

$$3 + 2 \times 5 =$$



should one do the addition first or the multiplication first? Try entering this expression into your calculator *exactly as it is written*. Do you get the answer 13? If so, your calculator knows the convention adopted by mathematicians everywhere that multiplication takes precedence over addition. The calculator has 'remembered' the 3 until it has worked out the result of multiplying 2 by 5 and has then added the 3 to the 10. According to the rules all mathematicians follow, if you wanted to add the 3 and the 2 first and then multiply that result by 5 you would have to write

$$(3 + 2) \times 5 = 25$$

Again, check that you can use the bracket function on your calculator to enter this expression exactly as written on the left-hand side of this equation and that you obtain the correct answer.

There are similar rules that govern the order of precedence of other arithmetic operations, which are neatly encapsulated in the mnemonic BEDMAS.

Order of arithmetic operations

Brackets take precedence over
Exponents. Then…
Division and
Multiplication must be done before…
Addition and
Subtraction.

So if we write $-3 - 12 \div 6$, the BEDMAS rules tell us we must do the division $(12 \div 6 = 2)$ before carrying out the subtraction $(-3 - 2 = -5)$. Try this on your calculator too; you may have to use the $+/-$ button to input the -3.

Many people, including scientists, find it hard to visualize the rules in a string of numbers. They often opt to use brackets to make things clear, even when those brackets simply reinforce the BEDMAS rules. So one could choose to write

$$(12 \div 3) + 2 = 6$$

There is nothing wrong with adding such ‘redundant’ brackets — they are simply there for clarity and can even be entered into your calculator (try it). Far better to have a few additional brackets than to be confused about the order in which the calculation must be carried out!

There is one final quirk associated with the use of brackets. In mathematics, the multiplication sign is often left out (though its presence is implied) between numbers and brackets, and between brackets and brackets. So

$$2(3+1) = 2 \times (3+1) = 8$$

and

$$(1+1)(4+3) = 2 \times 7 = 14$$

Some calculators ‘understand’ this convention and some do not. Check your own calculator carefully using the two examples above.

The next operation in precedence after brackets involves exponents. If there are powers in the expression you are evaluating, deal with any brackets first, then work out the powers before carrying out any other arithmetical operations.

Question

Evaluate 2×3^2 and $(2 \times 3)^2$

Answer

In the first case, there are no brackets so the exponent takes precedence:

$$2 \times 3^2 = 2 \times 9 = 18$$

In the second case, the bracket takes precedence:

$$(2 \times 3)^2 = 6^2 = 36$$

Question 1.12

Evaluate (preferably without using your calculator):

(a) $35 - 5 \times 2$ Answer

(b) $(35 - 5) \times 2$ Answer

(c) $5(2 - 3)$ Answer

(d) 3×2^2 Answer

(e) $2^3 + 3$ Answer

(f) $(2 + 6)(1 + 2)$ Answer

1.5 Learning outcomes for Chapter 1

After completing your work on this chapter you should be able to:

1.1 carry out addition, subtraction, multiplication and division operations involving negative numbers;

1.2 add two or more fractions;

1.3 subtract one fraction from another;

1.4 multiply a fraction by an integer or by another fraction;

1.5 divide a fraction by a non-zero integer or by another fraction;

1.6 evaluate powers involving any base and positive, negative or fractional exponents;

1.7 multiply or divide two powers involving the same base;

1.8 evaluate any given power of a number already raised to a power.

Measurement in Science

2

Observation, measurement and the recording of data are central activities in science. Speculation and the development of new theories are crucial as well, but ultimately the predictions resulting from those theories have to be tested against what actually happens and this can only be done by making further measurements. Whether measurements are made using simple instruments such as rulers and thermometers, or involve sophisticated devices such as electron microscopes or lasers, there are decisions to be made about how the results are to be represented, what units of measurements will be used and the precision to which the measurements will be made. In this chapter we will consider these points in turn. Then in Chapter 3 we will go on to think about how measurements of different quantities may be combined, and what significance should be attached to the results.

2.1 Large quantities and small quantities

Scientists frequently deal with enormous quantities — and with tiny ones. For example it is estimated that the Earth came into being about four and a half thousand million years ago. It took another six hundred million years for the first living things — bacteria — to appear. Bacteria are so small that they bear roughly the same proportion to the size of a pinhead as the size that pinhead bears to the height of a four-year old child!

In the previous chapter, we saw how convenient powers of ten could be as a way of writing down very large or very small numbers. For example,

$$10^6 = 1\,000\,000 \text{ (a million) and } 10^{-3} = 1/1000 = 0.001 \text{ (a thousandth)}$$

This shorthand can be extended to any quantity, simply by multiplying the power of ten by a small number. For instance,

$$2 \times 10^6 = 2 \times 1\,000\,000 = 2\,000\,000 \text{ (two million)}$$

(The quantity on the left-hand side would be said as ‘two times ten to the six’.)

Similarly,

$$3.5 \times 10^6 = 3\,500\,000 \text{ (three and a half million)}$$

$$7 \times 10^{-3} = 7/1000 = 0.007 \text{ (seven-thousandths)}$$

Scientists make so much use of this particular shorthand that it has come to be known as scientific notation (although in maths texts you may also find it referred to as standard index form or standard form.)

> A quantity is said to be expressed in scientific notation if its value is written as a number multiplied by a power of ten. The number can be a single digit or a decimal number, but must be greater than or equal to 1 and less than 10.

Note the restriction: 75×10^2 is not in scientific notation and nor is 0.75×10^4, though these are both equivalent to 7.5×10^3 which *is* in scientific notation.

Scientific notation can be defined more succinctly by making use of some of the mathematical symbols denoting the relative sizes of quantities. These symbols are:

$>$ greater than (e.g. $3 > 2$);

$\geq$ greater than or equal to (e.g. $a \geq 4$ means that the quantity a may take the exact value 4 or any value larger than 4);

$<$ less than;

$\leq$ less than or equal to.

Note that '$a \geq 4$' and '$4 \leq a$' convey exactly the same information about the quantity a.

Using these symbols, scientific notation may be defined as a notation in which the value of a quantity is written in the form $a \times 10^{n}$, where n is an integer and $1 \leq a < 10$.

To move from scientific notation to integers or to decimal notation, first deal with the power of ten, then carry out the multiplication or division.

Worked example 2.1

Express the following numbers as integers or in decimal notation:

(a) 4.53×10^3

(b) 8.371×10^2

(c) 6.4×10^{-3}

Answer

(a) $4.53 \times 10^3 = 4.53 \times 1000 = 4530$

(b) $8.371 \times 10^2 = 8.371 \times 100 = 837.1$

(c) $6.4 \times 10^{-3} = 6.4 \times \dfrac{1}{1000} = \dfrac{6.4}{1000} = 0.0064$

Note that, as in Worked example 2.1, a requirement to express a quantity in a different form simply involves taking the quantity and writing down its equivalent in the new form. You may do this in one step, or write down intermediate steps as was done in the worked example.

Question 2.1

Without using your calculator, express the following numbers as integers or in decimal notation. Note that (a) and (b) are in scientific notation, while (c) is not.

(a) 5.4×10^4 Answer

(b) 2.1×10^{-2} Answer

(c) 0.6×10^{-1} Answer

Moving from an integer or decimal notation to scientific notation is equivalent to deciding what power of ten you need to multiply or divide by in order to convert the number you are starting with to a number that lies between 1 and 10.

Worked example 2.2

Express the following numbers in scientific notation:

(a) 356 000

(b) 49.7×10^4

(c) 0.831

Answer

(a) $356\,000 = 3.56 \times 100\,000 = 3.56 \times 10^5$

(b) $49.7 \times 10^4 = 4.97 \times 10 \times 10^4 = 4.97 \times 10^{(1+4)} = 4.97 \times 10^5$

(c) $0.831 = \dfrac{8.31}{10} = 8.31 \times 10^{-1}$

In this worked example, all the steps have been written out in full. You may be able to manage with fewer steps in your own calculations — just use as many or as few as you feel comfortable with in order to get the right answer!

Question 2.2

Without using your calculator, express the following numbers in scientific notation:

(a) 215 Answer

(b) 46.7 Answer

(c) 152×10^3 Answer

(d) 0.000 0876 Answer

It is only too easy to lose track of the sizes of things when using scientific notation, so you should make a habit of thinking carefully about what the numbers mean, bearing in mind that numbers may be positive or negative. For example:

-1×10^{10} is a very large negative number;

-1×10^{-10} is a very small negative number;

1×10^{-10} is a very small positive number.

Figure 2.1 places on the number line some numbers in scientific notation. You may find this helps you to visualize things.

We started this section thinking about the early Earth and the first appearance of life. Using scientific notation, the age of the Earth can be neatly expressed as 4.6×10^9 years and the size of one type of those early bacteria as 1.2×10^{-6} metres. Of course the value we come up with for such sizes will depend on the units in which we choose to make the measurements. If we were measuring the diameter of the Moon, we could elect to express it in metres or in kilometres, or even in miles.

2.2 Units of measurement

In the UK, two systems of units are in common use. We still use old imperial measures for some things: milk is sold in pints and signposts indicate distances in miles. But for many other everyday measurements metric units have been adopted: we buy petrol in litres and sugar in kilogram bags. A great advantage of metric units is that we no longer have to convert laboriously from imperial units, such as gallons, feet and inches, in order to trade with continental Europe. Also, calculations are easier in a metric (i.e. decimal) system! Similar advantages were the main consideration when in 1960 an international conference formally approved a standard set of scientific units, thus replacing at a stroke the many different systems of measurement that had been used up until then by scientists of different nationalities. This 'universal' system for scientific measurement is referred to as SI units (short for Système International d'Unités).

In SI, there are seven ‘base units’, which are listed in Box 2.1. Surprising as it may initially seem, every unit for every other kind of quantity (speed, acceleration, pressure, energy, voltage, heat, magnetic field, properties of radioactive materials, indeed whatever you care to name) can be made up from combinations of just these seven base units. For instance, speed is measured in metres per second. You will find some other combinations of base units described in Chapter 3. In this course we shall work mainly with the familiar base units of length, mass, time and temperature, and some of their combinations, but it is worth knowing that the other base units exist as you may meet them in other courses.

Box 2.1 The SI base units

Physical quantity	Name of unit	Symbol for unit
length	metre	m
time	second	s
mass	kilogram	kg
temperature	kelvin	K
amount of substance	mole	mol
electric current	ampere	A
luminous intensity	candela	cd

Most of these base units relate to physical descriptions that apply universally. The SI base unit of time, the second, is defined as the period over which the waves emitted by caesium atoms under specific conditions cycle exactly 9192 631 770 times. Then the SI base unit of length, the metre, is defined by stating that the speed of light in a vacuum, which is a constant throughout the Universe, is exactly 299 792 458 metres per second.

The SI base unit of mass, the kilogram, is the only fundamental unit that is defined in terms of a specific object. The metal cylinder which constitutes the world's 'standard kilogram' is kept in France. Note that the kilogram is actually the standard unit of *mass*, not of *weight*. In scientific language, the weight of an object is the downward pull on that object due to gravity, whereas its mass is determined by the amount of matter in it. When astronauts go to the Moon, where the pull of gravity is only about one-sixth of that on Earth, their mass remains the same but their weight drops dramatically! And in zero gravity, they experience a condition known as 'weightlessness'.

The SI base unit of temperature is the kelvin, which is related to the everyday unit of temperature, the degree Celsius:

$$\text{(temperature in kelvin)} = \text{(temperature in degrees Celsius)} + 273.15$$

(You will find some of the rationale for the kelvin scale of temperature in Chapter 5.)

The amount of a pure substance is expressed in the SI base unit of the mole. Whatever the smallest particle of a given substance is, one mole of that substance will contain $6.02211367 \times 10^{23}$ (known as Avogadro's number) of those particles. A mole of graphite contains Avogadro's number of carbon atoms. Carbon dioxide is made up of molecules in which one carbon atom is joined to two oxygen atoms, and a mole of carbon dioxide contains Avogadro's number of these molecules.

You will have noticed that while the base unit of length is the metre, not the kilometre, the base unit of mass is the kilogram, not the gram.

It is important to realize that, although in everyday usage it is common to say that you 'weigh so many kilos', there are two things wrong with this usage from the scientific point of view. First, as noted in Box 2.1, the kilogram is not a unit of weight, but a unit of mass. (The SI unit of weight, the newton, will be discussed in Chapter 3.) Secondly, in scientific language, 'kilo' is never used as an abbreviation for kilogram, in the sense of the everyday phrase 'he weighs so many kilos'. In science, kilo is always used as a *prefix*, denoting a thousand: one kilometre is a thousand metres, one kilogram is a thousand grams.

Another prefix with which everybody is familiar is 'milli', denoting a thousandth. One millimetre, as marked on ordinary rulers, is one-thousandth of a metre; or put the other way round, a thousand millimetres make up a metre. There are many other prefixes in use with SI units, all of which may be applied to any quantity. Like kilo and milli, the standard prefixes are based on multiples of 1000 (i.e. 10^3). The most commonly used prefixes are listed in Box 2.2.

It is important to write the symbols for units and their prefixes in the correct case. So k (lower case) is the symbol for the prefix 'kilo' whilst K (upper case) is the symbol for the Kelvin; m (lower case) is the symbol for the metre or the prefix 'milli' whilst M (upper case) is the symbol for the prefix 'mega'.

Box 2.2 Prefixes used with SI units

prefix	symbol	multiplying factor
tera	T	10^{12} = 1000 000 000 000
giga	G	10^{9} = 1000 000 000
mega	M	10^{6} = 1000 000
kilo	k	10^{3} = 1000
–	–	10^{0} = 1
milli	m	10^{-3} = 0.001
micro	μ*	10^{-6} = 0.000 001
nano	n	10^{-9} = 0.000 000 001
pico	p	10^{-12} = 0.000 000 000 001
femto	f	10^{-15} = 0.000 000 000 000 001

* The Greek letter μ is pronounced 'mew'.

The following data may help to illustrate the size implications of some of the prefixes:

- the distance between Pluto (the furthest planet in the Solar System) and the Sun is about 6 Tm,
- a century is about 3 Gs,
- eleven and a half days contain about 1 Ms,
- the length of a typical virus is about 10 nm,
- the mass of a typical bacterial cell is about 1 pg.

Astronomers have long been making measurements involving very large quantities, but scientists are increasingly probing very small quantities. ‘Femtochemistry’ is a rapidly developing area, which involves the use of advanced laser techniques to investigate the act of chemical transformation as molecules collide with one another, chemical bonds are broken and new ones are formed. In this work, measurements have to be made on the femtosecond timescale. Ahmed H. Zewail (whose laboratory at the California Institute of Technology in Pasadena is often referred to as ‘femtoland’) received the 1999 Nobel Prize in Chemistry for his development of this new area.

Although scientific notation, SI units and the prefixes in Box 2.2 are universal shorthand for all scientists, there are a few instances in which other conventions and units are adopted by particular groups of scientists for reasons of convenience. For example, we have seen that the age of the Earth is about 4.6×10^9 years. One way to write this would be 4.6 'giga years' but geologists find millions of years a much more convenient standard measure. They even have a special symbol for a million years: Ma (where the 'a' stands for 'annum', the Latin word for year). So in Earth science texts you will commonly find the age of the Earth written as 4600 Ma. It won't have escaped your notice that the year is not the SI base unit of time — but then perhaps it would be a little odd to think about geological timescales in terms of seconds!

A few metric units from the pre-SI era also remain in use. In chemistry courses, you may come across the ångström (symbol Å), equal to 10^{-10} metres. This was commonly used for the measurement of distances between atoms in chemical structures, although these distances are now often expressed in either nanometres or picometres. Other metric but non-SI units with which we are all familiar are the litre (symbol l) and the degree Celsius (symbol °C).

There are also some prefixes in common use, which don't appear in Box 2.2 because they don't conform to the 'multiples of 1000' rule, but that when applied to particular units happen to produce a very convenient measure. One you will certainly have used yourself is centi (hundredth): rulers show centimetres (hundredths of a metre) as well as millimetres, and standard wine bottles are marked as holding 75 cl. One less commonly seen is deci (tenth), but that is routinely used by chemists in measuring concentrations of chemicals dissolved in water, or other solvents, as you will see in Chapter 3. In the next section you will also come across the decibel, which is used to measure the loudness of sounds.

Worked example 2.3

Diamond is a crystalline form of carbon in which the distance between adjacent carbon atoms is 0.154 nm. What is this interatomic distance expressed in picometres?

Answer

$$1 \text{ pm} = 10^{-12} \text{ m so}$$

$$1 \text{ m} = \frac{1}{10^{-12}} \text{ pm} = 10^{12} \text{ pm}$$

$$1 \text{ nm} = 10^{-9} \text{ m so}$$

$$\begin{aligned} 1 \text{ nm} &= 10^{-9} \times 10^{12} \text{ pm} \\ &= 10^{-9+12} \text{ pm} \\ &= 10^{3} \text{ pm} \end{aligned}$$

$$\begin{aligned} 0.154 \text{ nm} &= 0.154 \times 10^{3} \text{ pm} \\ &= 154 \text{ pm} \end{aligned}$$

Question 2.3

Using scientific notation, express:

(a) 3476 km (the radius of the Moon) in metres. Answer

(b) 8.0 μm (the diameter of a capillary carrying blood in the body) in nm, Answer

(c) 0.8 s (a typical time between human heartbeats) in ms. Answer

2.3 Scales of measurement

In thinking about the sizes of things, it is sometimes useful to do so in quite rough terms, just to the nearest power of ten. For example, 200 is nearer to 100 than it is to 1000, but 850 is nearer to 1000 than it is to 100. So if we were approximating to the nearest power of ten we could say 200 was roughly 10^2, but 850 was roughly 10^3. This process is called reducing the numbers to the nearest order of magnitude.

The approximate value of a quantity expressed as the nearest power of ten to that value is called the order of magnitude of the quantity.

The easiest way to work out the order of magnitude of a quantity is to express it first in scientific notation in the form $a \times 10^n$. Then if a is less than 5, the order of magnitude is 10^n. But if a is equal to or greater than 5, the power of ten is rounded up by one, so the order of magnitude is 10^{n+1}. For example, the diameter of Mars is 6762 km. This can be written as 6.762×10^3 km, and because 6.762 is greater than 5, the diameter of Mars is said to be 'of order 10^4 km'.

This is normally written as:

$$\text{diameter of Mars} \sim 10^4 \text{ km}$$

where the symbol $\sim$ denotes 'is of order'.

Question

What is the order of magnitude of the mass of the Earth, 6.0×10^{24} kg?

Answer

Mass of the Earth $\sim 10^{25}$ kg (since 6.0 is greater than 5, the power of ten has been rounded up).

Question

What is the order of magnitude of the mass of Jupiter, 1.9×10^{27} kg?

Answer

Mass of Jupiter $\sim 10^{27}$ kg (since 1.9 is less than 5, the power of ten remains unchanged).

Question

What is the order of magnitude of the average lifetime of unstable 'sigma plus' particles, 0.7×10^{-10} s?

Answer

$$\begin{aligned} \text{Particle lifetime} &= 0.7 \times 10^{-10}\ \text{s} \\ &= 7 \times 10^{-11}\ \text{s} \\ &\sim 10^{(-11+1)}\ \text{s} \\ &\sim 10^{-10}\ \text{s} \end{aligned}$$

Since 7 is greater than 5, the power of ten must be rounded up

The phrase ‘order of magnitude’ is also quite commonly used to compare the sizes of things, e.g. a millimetre is three orders of magnitude smaller than a metre.

Worked example 2.4

To the nearest order of magnitude, how many times more massive is Jupiter than the Earth?

Answer

We had:

$$\text{mass of Jupiter} \sim 10^{27}\ \text{kg}$$

and

$$\text{mass of Earth} \sim 10^{25}\ \text{kg}$$

so

$$\frac{\text{mass of Jupiter}}{\text{mass of Earth}} \sim \frac{10^{27}}{10^{25}} \sim 10^{(27-25)} \sim 10^{2}$$

Jupiter is two orders of magnitude (i.e. roughly 100 times) more massive than the Earth.

Question 2.4

What is the order of magnitude of the following measurements?

(a) The distance between Pluto (the furthest planet in the Solar System) and the Sun: five thousand nine hundred million kilometres. Answer

(b) The diameter of the Sun, given that its radius is 6.97×10^7 m. Answer

(c) 2π. Answer

(d) The mass of a carbon dioxide molecule: 7.31×10^{-26} kg. Answer

Sophisticated instrumentation now allows scientists to measure across 40 orders of magnitude, as shown in Figure 2.2. If you turn back to Figure 1.2, you will see that the scale there is quite different to that in Figure 2.2. On the thermometer, the interval between marked points was always the same, with marked points at $-0.1, 0, 0.1, 0.2$, etc. In other words, each step from one division to the next on the scale represented the *addition or subtraction* of a fixed amount (0.1 in that case). This kind of scale is called linear. In Figure 2.2, on the other hand, each step involves *multiplication or division* by a fixed power of ten (10^2 in this particular case). As a result, the intervals between divisions are all different. This kind of scale is called logarithmic. The next question allows you to investigate some of the properties of this type of scale.

Question 2.5

Use information from Figure 2.2 to answer the following questions.

(a) What is the difference in value between: Answer

(i) the tick marks at 10^{-2} m and 10^{0} m;

(ii) the tick marks at 10^{0} m and 10^{2} m, and

(iii) the tick marks at 10^{2} m and 10^{4} m?

(b) Calculate to the nearest order of magnitude, how many times taller than a child is Mount Everest. Answer

(c) Calculate to the nearest order of magnitude, how many typical viruses laid end to end would cover the thickness of a piece of paper. (*Hint*: you may find it helpful to look back at Worked example 2.4.) Answer

2.3.1 Logarithmic scales in practice

In Figure 2.2, a logarithmic scale was used for the purposes of display, and the power of ten for the multiplying factor (10^2) was chosen because it was the one that best fitted the page. In drawing diagrams and graphs we are always free to choose the scale divisions. However, logarithmic scales are used in a number of fields to measure quantities that can vary over a very wide range. In such cases, an increase or decrease of one 'unit' always represents a ten-fold increase or decrease in the quantity measured. The following sections give two examples.

Sound waves

The *decibel* (symbol dB) is the unit used to measure the relative loudness of sounds. The 'intensity' of a sound is related to the square of the variation in pressure as the sound wave passes through the air, and the range of intensities that people can detect is enormous. The sound that just causes pain is 10^{12} times more intense than the sound that is just audible! To deal with this huge range, a logarithmic scale for loudness was devised, according to which every 10 dB (or '1 B') increase in sound level is equivalent to a 10-fold increase in intensity. The decibel is also a convenient measure because a sound level of 1 dB is just within the limit of human hearing, and a change of 1 dB is about the smallest difference in sound that the ear can detect. (See Figure 2.3.)

Earthquakes

The *Richter scale* describes the magnitude of earthquakes. An instrument called a seismometer is used to measure the maximum ground movement caused by the earthquake, and a correction factor is applied to this reading to allow for the distance of the seismometer from the site of the earthquake. Seismometers are very sensitive and can detect minute amounts of ground movement (they have to be shielded from the effects caused just by people walking near them), but some earthquakes can produce ground movements millions of times greater than the minimum detectable limit. To cope with this huge variation, the Richter scale is logarithmic: an increase of one unit on the scale implies a ten-fold increase in the maximum ground movement. A magnitude 2 earthquake can just be felt as a tremor. A magnitude 3 earthquake produces 10 times more ground motion than a magnitude 2 earthquake. Damage to buildings occurs at magnitudes in excess of 6. The three largest earthquakes ever recorded (in Portugal in 1775, in Columbia in 1905 and in Japan in 1933) each had a Richter magnitudes of 8.9.

Worked example 2.5

A whisper corresponds to a sound level of about 20 dB, and a shout to a level of about 80 dB. How much greater is the intensity of a shout compared to that of a whisper?

Answer

The increase in sound level is

$$80 \text{ dB} - 20 \text{ dB} = 60 \text{ dB}$$

This may be expressed as (10 dB + 10 dB + 10 dB + 10 dB + 10 dB + 10 dB), and *each* 10 dB increase corresponds to multiplying the intensity by 10.

So the intensity of a shout is $(10 \times 10 \times 10 \times 10 \times 10 \times 10) = 10^6$ times greater than a whisper!

Question 2.6 Answer

How much more ground movement is there in an earthquake measuring 7 on the Richter scale compared to one measuring 3?

The basis of logarithmic scales will be discussed in Chapter 7.

2.4 How precise are the measurements?

Scientists are always trying to get better and more reliable data. One way of getting a more precise measurement might be to switch to an instrument with a more finely divided scale. Figure 2.4 shows parts of two thermometers placed side by side to record the air temperature in a room.

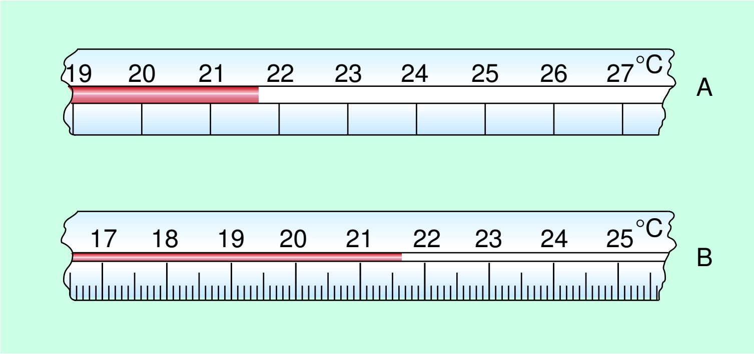


Figure 2.4: Parts of two thermometers A and B, measuring the air temperature in the same place.

The scale on thermometer A is quite coarse. The marked divisions represent integer numbers of degrees. On this scale we can see that the temperature is between 21 °C and 22 °C. I might estimate it as 21.7 °C, but somebody else could easily record it as 21.6 °C or 21.8 °C. So there is some uncertainty in the first decimal place, and certainly there is no way we could attempt to guess the temperature to two decimal places using this particular thermometer.

Thermometer B has a finer scale, with divisions marked every 0.1 °C. Now we can clearly see that the temperature is between 21.6 °C and 21.7 °C. I might read it as 21.63 °C, but a second person could plausibly read it as 21.61 °C or 21.65 °C. With this scale we are sure of the first decimal place but uncertain of the second.

When quoting the result of a measurement, you should never quote more digits than you can justify in terms of the uncertainty in the measurement. The number of significant figures in the value of a measured quantity is defined as the number of digits known with certainty plus one uncertain digit. With thermometer A we could be sure of the 21 (two digits), but were uncertain about the digit in the first decimal place, so we can quote a reading to three significant figures, as 21.7 °C (or 21.6 °C or 21.8 °C). With thermometer B it was the fourth digit that was uncertain, so we can quote our reading to four significant figures, as, for example, 21.64 °C.

Question 2.7 Answer

How many significant figures are quoted in each of the following quantities: 1221 m; 223.4 km; 1.487 km?

Question 2.7 emphasizes that significant figures mustn't be confused with the number of decimal places. After all, if you had measured the length of something as 13 mm, you wouldn't want the precision of your result to be changed just because you converted the measurement to centimetres. Whether you write 13 mm or 1.3 cm you are expressing the result of your measurement to two significant figures. Now suppose you convert to metres: 0.013 m. The uncertainty in your result still hasn't changed, so this shows that *leading zeroes in decimal numbers do not count as significant figures*. Scientific notation is helpful in this regard. Expressing the result as 1.3×10^{-2} m makes it very obvious that there are two significant figures.

Another circumstance in which one has to be careful about not using unjustified precision occurs when the results of measurements are used as the basis for calculations. Suppose we had measured the diameter of a circular pattern to two significant figures and obtained the result 3.3 cm. If we then needed to calculate the radius of the circle, it might be tempting simply to divide the diameter by 2 and say 'the radius of the pattern is 1.65 cm'. But 1.65 cm implies that the value is known to three significant figures! So we need to round off the figure in some way, to express the fact that the last significant digit in this particular case is the first digit after the decimal point. The usual rule for doing this is to leave the last significant digit unchanged if it would have been followed by a digit from 0 to 4, and to increase it by one if it would have been followed by a digit from 5 to 9. To two significant figures our circular pattern therefore has a radius of 1.7 cm. The issues involved in dealing with significant figures in more complex calculations are discussed in Chapter 3.

Scientific notation also shows up the need for care in dealing with very large numbers. The speed of light in a vacuum (the constant c in Einstein's equation $E = mc^2$ is, to six significant figures, 299 792 kilometres per second. Remembering the rounding rule, this can quite properly be written as 3×10^5 kilometres per second (one significant figure), or 3.00×10^5 kilometres per second (three significant figures). But it would be misleading to write it as 300 000 kilometres per second, because that could imply that all six digits are significant.

One of the advantages of using scientific notation is that it removes any ambiguity about whether zeroes at the *end* of a number are significant or are simply place markers. For example, if a length is measured to just one significant figure as 8 m, how should the equivalent value in centimetres be expressed? It would be misleading to write 800 cm, since that could imply the value is known to three significant figures. The only way out of this difficulty is to use scientific notation: writing 8×10^2 cm makes it clear that the quantity is known only to one significant figure, in line with the precision of the original measurement.

Question

If the speed of light through glass is quoted as 2.0×10^8 metres per second, how many significant figures are being given?

Answer

Final zeroes *are* significant, so the speed is being given to two significant figures.

Question

Neon gas makes up 0.0018% by volume of the air around us. How many significant figures are being given in this percentage?

Answer

Leading zeroes are *not* significant, so this value is also being given to two significant figures.

Worked example 2.6

The average diameter of Mars is 6762 km. What is this distance in metres, expressed to three significant figures?

Answer

The only way to express this quantity unambiguously to fewer than the four significant figures originally given is to use scientific notation.

$$\begin{aligned} 6762\ \text{km} &= 6.762 \times 10^{3}\ \text{km} \\ &= 6.762 \times 10^{3} \times 10^{3}\ \text{m} \\ &= 6.762 \times 10^{(3+3)}\ \text{m} \end{aligned}$$

Thus 6762 km = 6.762×10^6 m.

The final digit is a 2, so no rounding up is required and the average diameter of Mars is 6.76×10^6 m to three significant figures.

Question 2.8

Express the following temperatures to two significant figures:

(a) −38.87 °C (the melting point of mercury, which has the unusual property for a metal of being liquid at room temperature); Answer

(b) −195.8 °C (the boiling point of nitrogen, i.e. the temperature above which it is a gas); Answer

(c) 1083.4 °C (the melting point of copper). Answer

In the following chapter and in your future studies of science generally, you will be doing lots of calculations with numbers in scientific notation, and will also be expected to quote your results to appropriate numbers of significant figures. Chapter 3 will discuss the efficient way to input scientific notation into your calculator, and how to interpret the results.

2.5 Learning outcomes for Chapter 2

After completing your work on this chapter you should be able to:

2.1 convert quantities expressed as integers or in decimal notation to scientific notation and vice versa;

2.2 use prefixes in association with the SI base units and convert between prefixes;

2.3 express a given quantity as an order of magnitude;

2.4 state the number of significant figures in any given quantity;

2.5 express a given quantity to any stipulated number of significant figures.

3 Calculating in Science

There comes a point in science when simply measuring is not enough and we need to *calculate* the value of a quantity from values for other quantities that have been measured previously. Take, for example, the piece of granite shown in Figure 3.1. We can measure the lengths of its sides and its mass. With a little calculation we can also find its volume, its density, and the speed at which seismic waves will pass through a rock of this type following an earthquake.

This chapter looks at several scientific calculations, and in the process considers the role of significant figures, scientific notation and estimating when calculating in science. In addition, it introduces unit conversions and the use of formulae and equations.

Figure 3.1: A specimen of granite.

3.1 Calculating area; thinking about units and significant figures

Suppose we want to find the area of the top of the granite specimen shown in Figure 3.1. The lengths of its sides, measured in centimetres, are shown in Figure 3.2, and the area of a rectangle is given by

$$\text{area of rectangle} = \text{length} \times \text{width}$$

Thus the area of the top of the granite is

$$\text{area} = 8.4\ \text{cm} \times 5.7\ \text{cm}$$

Multiplying the two numbers together gives 47.88. However, if given as a value for the area, this would be incomplete and incorrectly stated for two reasons.

1 No units have been given.

2 The values for length and width which we've used are each given to two significant figures, but 47.88 is to *four* significant figures. This is too many.

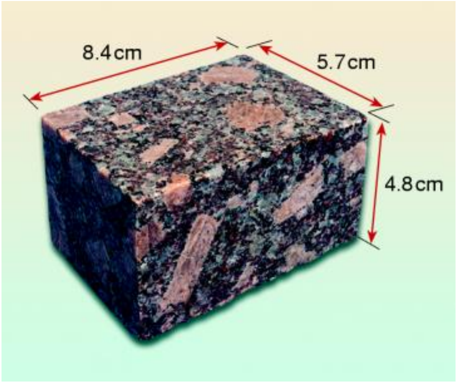


Figure 3.2: The lengths of the sides of the specimen of granite.

3.1.1 Units in calculations

The length and the width of the specimen of granite aren't just numbers, but physical quantities, with units. The area — the result of multiplying the length by the width — is a physical quantity too and it should also have units. The units which have been multiplied together are cm × cm, which can be written as $(\mathrm{cm})^2$, or more commonly as cm^2. In fact any unit of length squared will be a unit of area. Conversely, a value given for area should *always* have units of $(\mathrm{length})^2$.

All measurements should be given with appropriate units, and when performing calculations the units of the answer must always be consistent with the units of the quantities you input.

Care needs to be taken when multiplying together two lengths which have been measured in different units. Suppose, for instance, that we needed to find the area of a 1 cm by 4 m rectangle. Units of cm × m are meaningless; we need to convert the units to the same form before proceeding, and if in doubt it is best to convert to SI base units. Since 1 cm = 0.01 m, this gives an area of $0.01\ \mathrm{m} \times 4\ \mathrm{m} = 0.04\ \mathrm{m}^2$.

Question 3.1 Answer

Which of the following are units of area:

$(\text{inch})^2$; s^2; m^{-2}; cm^2; km^3; square miles?

Note: the symbols used for SI units are as given in Box 2.1.

3.1.2 Significant figures and rounding in calculations

It is not appropriate to quote answers to calculations to an unlimited number of significant figures. Suppose that, as part of a calculation, you were asked to divide 3.4 (known to two significant figures) by 2.34 (known to three significant figures). Entering $3.4 \div 2.34$ on most scientific calculators gives 1.452 991 453, but to quote a result to this number of significant figures would imply that you know the answer far more precisely than is really the case. The fact that 3.4 is quoted to two significant figures implies that the first digit is precisely known, but there is some uncertainty in the second digit; similarly the fact that 2.34 is quoted to three significant figures implies that there is some uncertainty in the third digit. Yet in giving the result as 1.452 991 453 we are claiming to be absolutely confident of the answer as far as 1.452 991 45, with just some uncertainty in the final digit. This is clearly nonsense!

The sensible number of significant figures to quote in any answer depends on a number of factors. However, in the absence of other considerations, a simple rule of thumb is useful:

> When multiplying and dividing numbers, the number of significant figures in the result should be the same as in the measurement with the *fewest* significant figures.

Applying this rule of thumb, the answer to the calculation $3.4 \div 2.34$ should be given to two significant figures, i.e. as 1.5.

Similarly, the result of the multiplication $8.4\ \text{cm} \times 5.7\ \text{cm}$ (used in finding the area of the top of the granite specimen) should be given as $48\ \text{cm}^2$, again to two significant figures.

There are two points of caution to bear in mind when thinking about the appropriate number of significant figures in calculations.

Avoiding rounding errors

You should round your answer to an appropriate number of significant figures at the end of a calculation. However, be careful not to round too soon, as this may introduce unnecessary errors, known as rounding errors. As an example of the dangers of rounding errors, let's return to our previous example. We found that:

$$3.4 \div 2.34 = 1.452\,991\,453$$

Or, giving the answer to two significant figures:

$$3.4 \div 2.34 = 1.5$$

Suppose that we now need to multiply the answer by 5.9:

$$1.452\,991\,453 \times 5.9 = 8.572\,649\,573 = 8.6 \text{ to two significant figures}$$

However, using the intermediate answer as quoted to two significant figures gives

$$1.5 \times 5.9 = 8.85 = 8.9 \text{ to two significant figures}$$

Rounding too soon has resulted in an incorrect answer.

The use of scientific calculators enables us to work to a large number of significant figures and so to avoid rounding errors. If this is not possible, you should follow the following advice:

> Work to at least one more significant figure than is required in the final answer, and just round at the end of the whole calculation.

In our example, the final answer should be given to two significant figures, which means that we should work using the result of the first calculation to at least three significant figures (1.45).

$$1.45 \times 5.9 = 8.555 = 8.6 \text{ to two significant figures.}$$

Applying common sense!

Always bear in mind the real problem that you are solving, and apply common sense in deciding how to quote the answer. Particular care needs to be taken when the calculation involves numbers which are *exactly* known. A light-hearted example should illustrate this point.

Question

Suppose you have 7 apples to share between 4 children. How many apples does each child get?

Answer

Dividing the number of apples by the number of children gives

$$\frac{7}{4} = 1.75$$

If we were to assume that the number of apples and number of children were each quoted to one significant figure, we would round the answer to one significant figure too, i.e. to 2 apples. But we would then need eight apples, which is more than we've got. In reality there are *exactly* 4 children and 7 apples, so the number of significant figures need not bother us. Provided we have a knife, it is perfectly possible to give each child 1.75 ($1\frac{3}{4}$) apples.

Question 3.2

Do the following calculations and express your answers to an appropriate number of significant figures.

(a) $\dfrac{6.732}{1.51}$ Answer

(b) 2.0×2.5 Answer

(c) $\left(\dfrac{4.2}{3.1}\right)^2$ Answer

(d) What is the total mass of three 1.5 kg bags of flour? Answer

3.2 Calculating in scientific notation

In science it is very often necessary to do calculations using very large and very small numbers, and scientific notation can be a tremendous help in this.

3.2.1 Calculating in scientific notation without a calculator

Suppose we need to multiply 2.50×10^4 and 2.00×10^5. The commutative nature of multiplication is completely general, so it applies when multiplying two numbers written in scientific notation too. This means that $(2.50\times10^4)\times(2.00\times10^5)$ can be written as $(2.50\times2.00)\times(10^4\times10^5)$, i.e.

$$\begin{aligned}(2.50\times10^4)\times(2.00\times10^5) &= (2.50\times2.00)\times(10^4\times10^5)\\ &= 5.00\times10^{4+5}\\ &= 5.00\times10^9\end{aligned}$$

All of the rules for the manipulation of powers discussed in Chapter 1 can be applied to numbers written in scientific notation, but care needs to be taken to treat the decimal parts of the numbers (such as the 2.50 in 2.50×10^5) and the powers of ten separately. So, for example

$$\frac{2.50\times10^4}{2.00\times10^5} = \frac{2.50}{2.00}\times\frac{10^4}{10^5} = \frac{2.50}{2.00}\times10^{4-5} = 1.25\times10^{-1}$$

and

$$\left(2.50\times10^5\right)^2 = 2.50^2\times\left(10^5\right)^2 = 6.25\times10^{10}$$

Question 3.3

Evaluate the following without using a calculator, giving your answers in scientific notation.

(a) $(3.0 \times 10^6) \times (7.0 \times 10^{-2})$ Answer

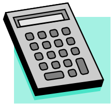
(b) $\dfrac{8 \times 10^4}{4 \times 10^{-1}}$ Answer

(c) $\dfrac{10^4 \times (4 \times 10^4)}{1 \times 10^{-5}}$ Answer

(d) $\left(3.00 \times 10^8\right)^2$ Answer

3.2.2 Using a calculator for scientific notation

In the rest of this chapter, and in your future studies of science generally, you will be doing many calculations with numbers in scientific notation, so it is very important that you know how to input them into your calculator efficiently and how to interpret the results.



First of all make sure that you can input numbers in scientific notation into your calculator.You can do this using the button you used to input powers in Section 1.3.1, but it is more straightforward to use the special button provided for entering scientific notation. This might be labelled as EXP, EE, E or EX, but there is considerable variation between calculators. Make sure that you can find the appropriate button on your calculator. Using a button of this sort is equivalent to typing the whole of ‘$\times 10$ to the power’. So, on a particular calculator, keying 2.5 EXP 12 enters the whole of 2.5×10^{12}.

In addition to being able to enter numbers in scientific notation into your calculator, it is important that you can understand your calculator display when it gives an answer in scientific notation.

Enter the number 2.5×10^{12} into your calculator and look at the display.

Again there is considerable variation from calculator to calculator, but it is likely that the display will be similar to one of those shown in Figure 3.3. The 12 at the right of the display is the power of ten, but notice that *the ten itself is frequently not displayed*. If your calculator is one of those which displays 2.5×10^{12} as shown in Figure 3.3e, then you will need to take particular care; this *does not* mean 2.5^{12} on this occasion. You should be careful not to copy down a number displayed in this way on your calculator as an answer to a question; this could cause confusion at a later stage.

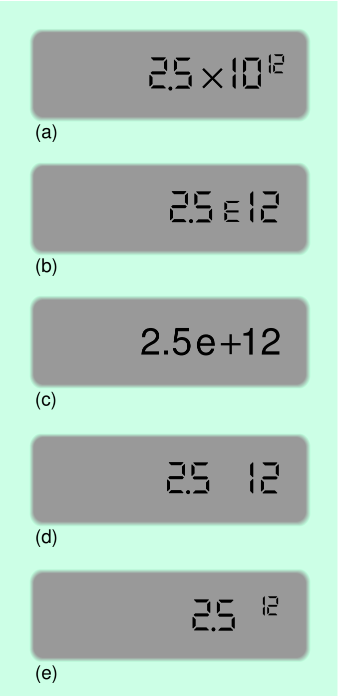


Figure 3.3: Examples of how various calculators would display the number 2.5×10^{12}

No matter how scientific notation is entered and displayed on your calculator or computer, when writing it on paper you should always use the form exemplified by 2.5×10^{12}.

To enter a number such as 5×10^{-16} into your calculator, you may need to use the button labelled something like +/− (as used in Section 1.1.3) in order to enter the negative exponent.

To enter a number such as 10^8 into your calculator using the scientific notation button, it can be helpful to remember that 10^8 is written as 1×10^8 in scientific notation, so you will need to key something like 1 EXP 8.

If you are at all unsure about using your calculator for calculations involving scientific notation, you should repeat Question 3.3, this time using your calculator.

Question 3.4 Answer

A square integrated circuit, used as the processor in a computer, has sides of length 9.78×10^{-3} m. Give its area in m^2 in scientific notation and to an appropriate number of significant figures.

3.3 Estimating answers

The first time I attempted Question 3.4, my calculator gave me the answer 95.6 m^2. This is incorrect (I'd forgotten to enter the power of ten). It is sensible to get into the habit of checking that the answer your calculator gives is reasonable, by estimating the likely answer. In the case of Question 3.4, the answer should be *approximately* $\left(1 \times 10^{-2}\ \text{m}\right)^2$ which you can see (without using a calculator!) is $1 \times 10^{-4}\ \text{m}^2$. So a calculator answer of 95.6 m^2 is clearly wrong.

In addition to being useful as a way of checking calculator answers, estimated answers are, in their own right, quite frequently all that is needed. Chapter 2 began with a comparison between the size of a bacterium and the size of a pinhead. We could use precise measuring instruments to find that the diameter of a particular bacterium is 1.69 μm (i.e. 1.69×10^{-6} m) and that the diameter of the head of a particular pin is 9.86×10^{-4} m. The diameter of the pinhead would then be

$$\frac{9.86 \times 10^{-4}\ \text{m}}{1.69 \times 10^{-6}\ \text{m}} = 5.83 \times 10^{2} \text{ times bigger than that of the bacterium.}$$

However, to get a feel for the relative sizes, we only really need to estimate the answer. If an estimate is all that is required, it is perfectly acceptable to work to one significant figure throughout (indeed, working to the nearest order of magnitude is sometimes sufficient) and since the final answer is only approximately known, the symbol '≈' (meaning 'approximately equal to') is used in place of an equals sign.

Worked example 3.1

Working to one significant figure throughout, estimate how many times bigger a pinhead of diameter 9.86×10^{-4} m is than a bacterium of diameter 1.69×10^{-6} m.

Answer

Diameter of pinhead $\approx 1 \times 10^{-3}$ m.
Diameter of bacterium $\approx 2 \times 10^{-6}$ m.

$$\begin{aligned}
\frac{\text{diameter of pinhead}}{\text{diameter of bacterium}} &\approx \frac{1 \times 10^{-3}\ \text{m}}{2 \times 10^{-6}\ \text{m}} \\
&\approx \frac{1}{2} \times \frac{10^{-3}}{10^{-6}} \\
&\approx 0.5 \times 10^{-3-(-6)} \\
&\approx 0.5 \times 10^{3} \\
&\approx 5 \times 10^{2}
\end{aligned}$$

So the diameter of the pinhead is approximately 500 times that of the bacterium.

It is important that you write out your mathematical calculations carefully, and one of the functions of the worked examples scattered throughout the course is to illustrate how to do this. There are three particular points to note from Worked example 3.1.

Taking care when writing maths

1 Note that the symbols $=$ and $\approx$ mean 'equals' and 'approximately equals' and should *never* be used to mean 'thus' or 'therefore'. It is acceptable to use the symbol $\therefore$ for 'therefore'; alternatively don't be afraid to write *words* of explanation in your calculations.

2 It can make a calculation clearer if you align the $=$ or $\approx$ symbols vertically, to indicate that the quantity on the left-hand side is equal to or approximately equal to each of the quantities on the right-hand side.

3 Note that the diameter of the bacterium and the pinhead each have metres (m) as their units, so when one diameter is divided by the other, the units cancel to leave a number with no units.

The handling of units in calculations is discussed further in Section 3.5.4.

Question 3.5 Answer

The average distance of the Earth from the Sun is 1.50×10^{11} m and the distance to the nearest star other than the Sun (Proxima Centauri) is 3.99×10^{16} m. Working to one significant figure throughout, estimate how many times further it is to Proxima Centauri than to the Sun.

3.4 Unit conversions

In calculating the area of the top of the granite specimen earlier in this chapter, we measured the length of the sides in centimetres and hence calculated the area in cm^2. If we had wanted the area in the SI units of m^2 we could have converted the lengths from centimetres to metres before starting the calculation. We would then have had

$$\text{area} = (8.4 \times 10^{-2}\ \text{m}) \times (5.7 \times 10^{-2}\ \text{m}) = 4.8 \times 10^{-3}\ \text{m}^2$$

It is best, whenever possible, to convert all units to SI units before starting on a calculation.

Unfortunately it is not always possible to convert units before commencing a calculation; sometimes you will be given an area in, say, cm^2, without knowing how the area was calculated, and you will need to convert this to an area in m^2. This section discusses this, as well as some more complex unit conversions.

3.4.1 Converting units of area

Let's start with an example which is relatively easy to visualize. Suppose we want to know how many mm^2 there are in a cm^2. There are 10 millimetres in a centimetre, so each side of the square centimetre in Figure 3.4 measures either 1 cm or 10 mm. To find the area, we need to multiply the length by the width. Working in centimetres gives

$$\text{area} = 1\ \text{cm} \times 1\ \text{cm} = (1\ \text{cm})^2 = 1^2\ \text{cm}^2 = 1\ \text{cm}^2$$

Working in millimetres gives

$$\text{area} = 10\ \text{mm} \times 10\ \text{mm} = (10\ \text{mm})^2 = 10^2\ \text{mm}^2 = 100\ \text{mm}^2$$

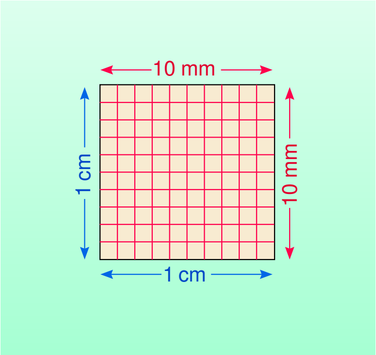


Figure 3.4: A square centimetre (not to scale)

Thus $1\ \text{cm}^2 = 100\ \text{mm}^2$ and $1\ \text{mm}^2 = \dfrac{1}{100}\ \text{cm}^2$.

If we want to convert from cm^2 to mm^2 we need to multiply by 100; if we want to convert from mm^2 to cm^2 we need to divide by 100.

Figure 3.5 illustrates another example which is a little harder to visualize. Each side of the square measures either 1 km or 1000 m (10^3 m). Working in kilometres gives

$$\text{area} = 1\text{ km} \times 1\text{ km} = (1\text{ km})^2 = 1^2\text{ km}^2 = 1\text{ km}^2$$

Working in metres gives

$$\text{area} = 10^3\text{ m} \times 10^3\text{ m} = \left(10^3\text{ m}\right)^2 = \left(10^3\right)^2\text{ m}^2 = 10^6\text{ m}^2$$

Thus $1\text{ km}^2 = 10^6\text{ m}^2$ and $1\text{ m}^2 = \dfrac{1}{10^6}\text{ km}^2$.

To convert from km^2 to m^2 we need to multiply by 10^6; to convert from m^2 to km^2 we need to divide by 10^6.

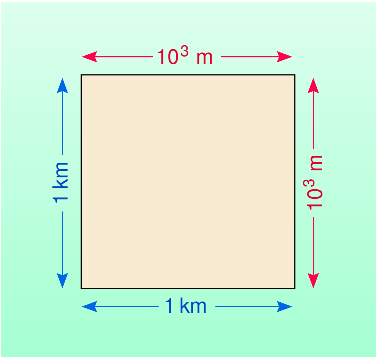


Figure 3.5: A square kilometre

The number by which we need to divide or multiply to convert from one unit to another is known as the ‘conversion factor’. In general, to convert between units of area we need to *square* the conversion factor which we would use to convert corresponding lengths.

As a final example consider a conversion between km^2 and mm^2.

There are 10^3 millimetres in a metre and 10^3 metres in a kilometre, so there are 10^6 millimetres in a kilometre as illustrated in Figure 3.6.

To convert from kilometres to millimetres we need to multiply by 10^6; however to convert from km^2 to mm^2 we need to multiply by $\left(10^6\right)^2$, i.e. 10^{12}.

Similarly, to convert from mm^2 to km^2 we need to divide by $\left(10^6\right)^2$, i.e. 10^{12}.

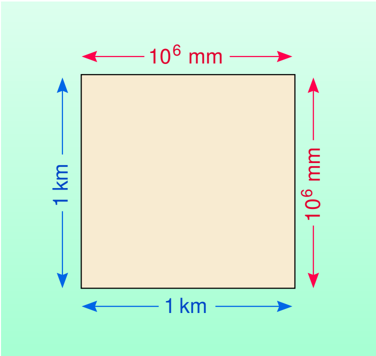


Figure 3.6: A square kilometre

Question 3.6

A desk has an area of 1.04 m^2. Express this area in:

(a) cm^2 Answer

(b) µm^2 Answer

(c) km^2 Answer

3.4.2 Converting units of volume

The volume of the piece of granite shown in Figure 3.2 is given by

$$\text{volume} = \text{length} \times \text{width} \times \text{height}$$

The lengths of the sides are 8.4 cm, 5.7 cm and 4.8 cm, so

$$\begin{aligned}\text{volume} &= 8.4\ \text{cm} \times 5.7\ \text{cm} \times 4.8\ \text{cm} \\ &= 2.3 \times 10^2\ \text{cm}^3 \text{ to two significant figures.}\end{aligned}$$

Note that the units which have been multiplied together are cm × cm × cm, so in this case the units of volume are cm^3. A value given for volume should *always* have units equivalent to those used for $(\text{length})^3$, and if we had converted the lengths of the sides to metres before doing the calculation, we would have obtained a value for volume in m^3:

$$\begin{aligned}\text{volume} &= (8.4 \times 10^{-2}\ \text{m}) \times (5.7 \times 10^{-2}\ \text{m}) \times (4.8 \times 10^{-2}\ \text{m}) \\ &= 2.3 \times 10^{-4}\ \text{m}^3 \text{ to two significant figures.}\end{aligned}$$

The method for converting between different units of volume is a direct extension of the method for converting between different units of area. Suppose we want to know how many mm^3 there are in a cm^3.

There are 10 mm in 1 cm, so each side of the cubic centimetre in Figure 3.7 measures either 1 cm or 10 mm. The volume can be written as either 1 cm^3 or 10^3 mm^3. Thus 1 $cm^3 = 10^3$ mm^3 and 1 $mm^3 = \frac{1}{10^3}$ cm^3. To convert from cm^3 to mm^3 we need to multiply by 10^3; to convert from mm^3 to cm^3 we need to divide by 10^3.

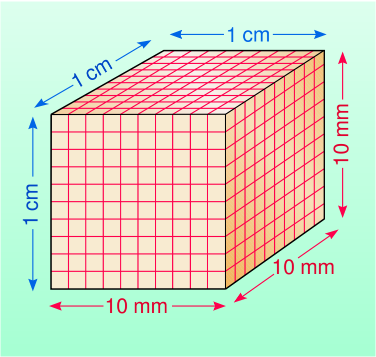


Figure 3.7: A cubic centimetre (not to scale).

In general, to convert between units of volume we need to *cube* the conversion factor that we would use to convert corresponding lengths.

We can convert a volume of 2.3×10^2 cm^3 into m^3 simply by saying that there are 10^2 cm in 1 m; hence there are $\left(10^2\right)^3$ cm^3 in 1 m^3, so

$$1\ \text{cm}^3 = \frac{1}{(10^2)^3}\ \text{m}^3$$

and

$$\begin{aligned} 2.3 \times 10^2\ \text{cm}^3 &= \frac{2.3 \times 10^2}{(10^2)^3}\ \text{m}^3 \\ &= 2.3 \times 10^{-4}\ \text{m}^3 \end{aligned}$$

This value is, of course, the same as the one we obtained from first principles!

The prefix ‘deci’ meaning one tenth was introduced in Section 2.2, thus 1 decimetre (dm) is one tenth of a metre. The cubic decimetre (dm^3) is sometimes used as a unit of volume. The litre (l) (also introduced in Chapter 2) was defined in 1901 as the volume of a kilogram of water at 4 °C, under standard atmospheric pressure. This volume turns out to be 1.000 28 dm^3, and since 1969 a litre has been *defined* to be 1 dm^3.

Worked example 3.2

Convert a volume of 1 dm^3 to: (a) cm^3 (b) m^3

Answer

(a) 1 m = 10 dm and 1 m = 100 cm so 1 dm = 10 cm.

Thus 1 $dm^3 = 10^3$ cm^3.

(b) 1 m = 10 dm

Thus 1 $m^3 = 10^3$ dm^3

and $1\ dm^3 = \frac{1}{10^3}\ m^3 = 10^{-3}\ m^3$.

{Thus 1 dm^3 (i.e. 1 litre) is a thousand times bigger than a cubic centimetre and a thousand times smaller than a cubic metre. You may already have been aware that 1 litre = 1000 cm^3. Thus 1 ml = 1 cm^3. }

Figure 3.8 is a summary of unit conversions for length, area and volume, but you should try to remember the general principles involved rather than memorizing individual conversion factors.

Question 3.7

Express each of the following volumes in scientific notation in m^3:

(a) the volume of the planet Mars, which is 1.64×10^{11} km^3; Answer

(b) the volume of a ball bearing, which is 16 mm^3. Answer

3.4.3 Converting units of distance, time and speed

You were introduced in Box 2.1 to the metre as the base unit of distance or length and to the second as the base unit of time. The average speed with which an object moves is the total distance travelled divided by the total time taken, so when Marion Jones won the women's 100-metre final at the 2000 Sydney Olympics in 10.75 s, her average speed was

$$\text{average speed} = \frac{100.0\ \text{m}}{10.75\ \text{s}} = 9.302\ \text{m}\,\text{s}^{-1}$$

Similarly, if a girl grows a total of 116 cm in 12.5 years, her average rate of growth is

$$\text{growth rate} = \frac{116\ \text{cm}}{12.5\ \text{years}} = 9.28\ \text{cm year}^{-1}$$

Note that it is appropriate to give the answer to the first example to four significant figures (assuming that the length of the running track was known to at least four significant figures). Also note the way in which the units have been written in both examples.

> The notation of negative exponents, which we have used to represent numbers like $1/2^3$ as 2^{-3} and $1/10^8$ as 10^{-8}, can also be used for units. So 1/s can be written as s^{-1}, m/s can be written as $m\,s^{-1}$ and cm/year can be written as $cm\,year^{-1}$.

The SI unit of speed is $m\,s^{-1}$ and this is usually said as 'metres seconds to the minus one'. Although $m\,s^{-1}$ is the correct scientific way of writing the unit, it is sometimes written as m/s, and quite frequently said as 'metres per second', even when written as $m\,s^{-1}$. The '/' for per is quite commonly used in other units too.

Many things move and/or grow in the world around us, and it is useful to compare different values for speed or rate of growth. Different speeds are frequently measured in different units, so in order to be able to compare like with like it is necessary to convert between different units for distance, time and speed. Box 3.1 considers various examples of speed and growth, and the text immediately following the box looks at ways of converting one unit to another.

Box 3.1 How fast?

Light (and other forms of radiation such as X-rays and radio waves) travels in a vacuum with a constant speed of 3.00×10^8 m s^{-1}. It is currently believed that nothing can travel faster than this.

Towards the opposite extreme are stalactites and stalagmites, which grow just fractions of a millimetre each year. A typical growth rate is 0.1 mm year^{-1}. Stalactites form when water drips from the roof of an underground cave, depositing calcite (frequently from the limestone in the rock above the cave) in an icicle shaped formation as it does so. Stalagmites form as the water drips onto the floor of the cave, depositing further calcite.

Figure 3.9: The Saskatchewan Glacier, Banff National Park, Canada.

It is not normally possible to detect the motion of a glacier by eye, but there is considerable variation in the speed with which they move. The Franz Josef Glacier in New Zealand is particularly fast moving, with an average speed of about 1.5 m day^{-1}. The speed of the Saskatchewan Glacier in Canada (Figure 3.9) is rather more typical, at about 12 cm day^{-1}.

In addition to geological processes such as glacier flow and stalactite formation, the theory of plate tectonics tells us that the surface of the Earth is itself moving.

The Earth's surface is thought to comprise seven major tectonic plates and numerous smaller ones, each only about 100 km thick but mostly thousands of kilometres in width. Evidence, including evidence from sea-floor spreading (to be discussed in Chapter 5) indicates that plates move relative to one another with speeds between about 10 km Ma^{-1} and 100 km Ma^{-1} (where Ma is the abbreviation for a million years, as discussed in Section 2.2).

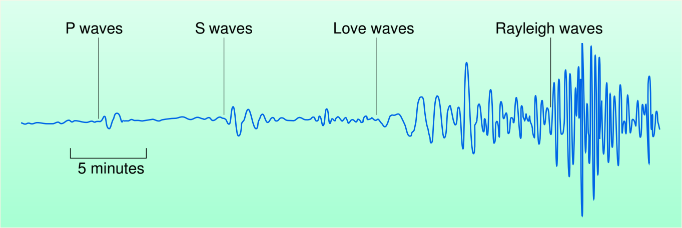


Figure 3.10: A seismogram (the printout from a seismometer) showing the arrival of P waves, S waves, Love waves and Rayleigh waves from a distant earthquake. Elapsed time increases from left to right.

Earthquakes and volcanoes occur all over the Earth, but they are more common close to the boundaries of tectonic plates than elsewhere. Following an earthquake, seismic waves (the word 'seismic' is from the Greek for 'shaking') travel out from the centre of the quake and are recorded by seismometers at various locations. There are several different types of seismic waves, including P waves, S waves, Love waves and Rayleigh waves, each travelling at different speeds (and sometimes also by different routes), so reaching a given seismometer at different times (see Figure 3.10). P waves travel fastest, with an average speed of about 5.6 km s^{-1} in rocks close to the Earth's surface, so reach the seismometer first (the name P wave was originally an abbreviation for primary wave). S waves (S for secondary) travel with an average speed of about 3.4 km s^{-1} in rocks close to the Earth's surface.

Perhaps the most dangerous sort of volcanic eruption is one that leads to a high-speed pyroclastic flow (a mixture of rock fragments and gases, moving as a fluid) away from the volcano. Pyroclastic flows are particularly destructive both because of their high temperatures (typically between 200 °C and 700 °C) and the high speed at which they travel (up to about 100 km hour^{-1}).

The speeds given so far have related to processes on the Earth, but remember that the Earth itself is moving too! The rotation of the Earth on its axis leads to a movement of up to 0.5 km s^{-1} at the surface. In addition, the Earth is orbiting the Sun at about 30 km s^{-1} and the entire Solar System is moving around the centre of the galaxy at about 250 km s^{-1}.

To convert from one unit of speed to another, we may need to convert both the unit of distance and the unit of time. To start with, let's consider the rather more straightforward case when we only have to convert the unit of distance, for example in converting from mm s^{-1} to m s^{-1}.

We know that $1\ \text{m} = 10^3\ \text{mm}$

$$\text{so } 1\ \text{mm} = \frac{1}{10^3}\ \text{m} = 1 \times 10^{-3}\ \text{m}$$

We can therefore say straight away that $1\ \text{mm s}^{-1} = 1 \times 10^{-3}\ \text{m s}^{-1}$

We have simply applied the same conversion factor as in converting from mm to m. Note that the answer makes sense: it is reasonable to expect that the numerical value of a speed in $\mathrm{m\,s^{-1}}$ will be smaller than the same speed when given in $\mathrm{mm\,s^{-1}}$.

Worked example 3.3

Convert the speed of the Earth as it orbits the Sun (given above as $30\ \mathrm{km\,s^{-1}}$) into a value in $\mathrm{m\,s^{-1}}$.

Answer

$$1\ \mathrm{km} = 1 \times 10^3\ \mathrm{m}$$

So

$$1\ \mathrm{km\,s^{-1}} = 1 \times 10^3\ \mathrm{m\,s^{-1}}$$

$$\begin{aligned} 30\ \mathrm{km\,s^{-1}} &= 30 \times 10^3\ \mathrm{m\,s^{-1}} \\ &= 3.0 \times 10^4\ \mathrm{m\,s^{-1}} \text{ in scientific notation.} \end{aligned}$$

The Earth orbits the Sun with a speed of about $3.0 \times 10^4\ \mathrm{m\,s^{-1}}$. Again the answer makes sense: it is reasonable to expect that the numerical value of a speed in $\mathrm{m\,s^{-1}}$ will be larger than the same speed when given in $\mathrm{km\,s^{-1}}$.

Next let's consider what happens when we need to convert only the time part of units of speed, for instance in converting from km hour^{-1} to km s^{-1}.

We know that there are 60 minutes in an hour and 60 seconds in a minute, so

$$1 \text{ hour} = 60 \times 60 \text{ s} = 3600 \text{ s}$$

However, in this case we don't want to convert from hours to seconds, but rather from kilometres *per hour* to kilometres *per second*. The way forward comes in recognizing that the word 'per' and the use of negative exponents in hour^{-1} and s^{-1} indicate division. So to convert from hour^{-1} to s^{-1} (or from km hour^{-1} to km s^{-1}) we need to find the conversion factor from hours to seconds and then *divide* by it.

$$1 \text{ hour} = 3600 \text{ s}$$

$$\text{so } 1 \text{ km hour}^{-1} = \frac{1}{3600} \text{ km s}^{-1}$$

In deciding whether to divide or multiply by a particular conversion factor, common sense can also come to our aid. It is reasonable to expect that a speed quoted in km s^{-1} will be *smaller* than the same speed when quoted in km hour^{-1}, so it is reasonable to *divide* by the 3600 on this occasion.

Worked example 3.4

Two tectonic plates are moving apart at an average rate of $35\ \text{km Ma}^{-1}$. Convert this to a value in km year^{-1}.

Answer

We know that

$$1\ \text{Ma} = 10^6\ \text{years}$$

so

$$1\ \text{km Ma}^{-1} = \frac{1}{10^6}\ \text{km year}^{-1}$$

and therefore

$$\begin{aligned} 35\ \text{km Ma}^{-1} &= \frac{35}{10^6}\ \text{km year}^{-1} \\ &= 3.5 \times 10^{-5}\ \text{km year}^{-1} \text{ in scientific notation.} \end{aligned}$$

The plates are moving apart at an average rate of $3.5 \times 10^{-5}\ \text{km year}^{-1}$.

This answer is reasonable: you would expect the rate of separation quoted in km year^{-1} to be smaller than the same rate quoted in km Ma^{-1}.

Question 3.8

Convert the average speed of the Saskatchewan Glacier ($12\ \text{cm day}^{-1}$) to a value in:

(a) m day^{-1} Answer

(b) cm s^{-1} Answer

Finally we need to consider conversions for speed in which both the units of distance and the units of time have to be converted. This is simply a combination of the techniques illustrated in Worked examples 3.3 and 3.4. Suppose we want to convert from km hour^{-1} to m s^{-1}.

$$1\ \text{km} = 10^3\ \text{m}$$
$$1\ \text{hour} = 3600\ \text{s}$$

To convert from km hour^{-1} to m s^{-1}, we need to *multiply* by 10^3 (to convert the km to m) and *divide* by 3600 (to convert the hour^{-1} to s^{-1}):

$$1\ \text{km hour}^{-1} = \frac{10^3}{3600}\ \text{m s}^{-1} = 0.278\ \text{m s}^{-1}\ \text{to three significant figures.}$$

Worked example 3.5

Convert the average speed of separation of the tectonic plates discussed in Worked example 3.4 (35 km Ma^{-1}) to a value in mm year^{-1}.

Answer

1 km = 10^3 m and 1 m = 10^3 mm, so 1 km = 10^6 mm

1 Ma = 10^6 year

To convert from km Ma^{-1} to mm year^{-1}, we need to *multiply* by 10^6 (to convert the km to mm) and *divide* by 10^6 (to convert the Ma^{-1} to year^{-1}.

$$1\ \text{km Ma}^{-1} = \frac{10^6}{10^6}\ \text{mm year}^{-1} = 1\ \text{mm year}^{-1}$$

Thus a speed given in km Ma^{-1} is numerically equal to one given in mm year^{-1}. The plates are moving apart at a 35 mm year^{-1}. This is similar to the rate at which human fingernails grow and is easier to imagine than is 35 km Ma^{-1}.

Question 3.9

Convert each of the following to values in $\mathrm{m\,s^{-1}}$ and then compare them.

(a) A stalactite growth rate of 0.1 $\mathrm{mm\,year^{-1}}$. Answer

(b) The average speed of the Saskatchewan Glacier (12 $\mathrm{cm\,day^{-1}}$). Answer

(c) The speed of separation of the tectonic plates discussed in Worked examples 3.4 and 3.5 (35 $\mathrm{km\,Ma^{-1}}$). Answer

(*Note*: for the purposes of this question, consider 1 year to be 365 days long.)

3.4.4 Concentration and density; more unit conversions

Methods for converting units for physical quantities, such as concentration and density, follow directly from the discussion in the previous sections.

Box 3.2 Concentration

The concentration of a solution is a term used as a measure of how much of a certain substance the solution contains, relative to the solution's total volume. For example, we may want to know how much sugar has been dissolved in water to give one litre of syrup.

The amount of the substance can be measured in moles, in which case the concentration will have units of mol l^{-1} or mol dm^{-3}. Alternatively, the amount can be measured by mass, in kg, g, mg, etc., leading to units for concentration of kg dm^{-3}, g m^{-3}, or mg l^{-1}, and so on.

The World Health Organization (WHO) sets limits for safe concentrations of various impurities in water, for example, the limit for the concentration of nitrates in water is currently $50\ \text{mg l}^{-1}$. This means that there should be no more than 50 mg of nitrate in each litre (dm^3) of water.

To convert a concentration from, say, $\mathrm{mg\,l^{-1}}$ to $\mathrm{\mu g\,ml^{-1}}$ you need to follow a very similar procedure to the one introduced in Section 3.4.3, as the following worked example shows.

It is very easy to confuse the letter ‘l’, used as the symbol for litres, with the number 1. Take care!

Worked example 3.6

Convert 50 $\mathrm{mg\,l^{-1}}$ (the World Health Organization’s limit for the concentration of nitrates in water) to a value in $\mathrm{\mu g\,ml^{-1}}$.

Answer

We can easily write down the conversion factors for mg to μg and from litres to ml.

$$1\ \mathrm{mg} = 10^3\ \mathrm{\mu g}$$

$$1\ \mathrm{litre} = 1\ \mathrm{l} = 10^3\ \mathrm{ml}$$

So to convert from $\mathrm{mg\,l^{-1}}$ to $\mathrm{\mu g\,ml^{-1}}$, we need to *multiply* by 10^3 (to convert the mg to μg) and *divide* by 10^3 (to convert the $\mathrm{l^{-1}}$ to $\mathrm{ml^{-1}}$).

$$1\ \mathrm{mg\,l^{-1}} = \frac{10^3}{10^3}\ \mathrm{\mu g\,ml^{-1}} = 1\ \mathrm{\mu g\,ml^{-1}}$$

Thus a concentration given in $\mathrm{mg\,l^{-1}}$ is numerically equal to one given in $\mathrm{\mu g\,ml^{-1}}$, in particular 50 $\mathrm{mg\,l^{-1}}$ = 50 $\mathrm{\mu g\,ml^{-1}}$.

Box 3.3 Density

The density of a piece of material is found by dividing its mass by its volume. In other words

$$\text{density} = \frac{\text{mass}}{\text{volume}}$$

If mass is measured in kg and volume is in m^3, then it follows that the unit of density will be kg/m^3 (said as 'kilograms per metre cubed') or, written in the form favoured in this course, $kg\,m^{-3}$ (said as 'kilograms metres to the minus three').

The density of pure water is $1 \times 10^3\ kg\,m^{-3}$; materials with a density greater than this (such as steel of density $7.8 \times 10^3\ kg\,m^{-3}$) will sink in water whereas materials of lower density (such as wood from an oak tree, density $6.5 \times 10^2\ kg\,m^{-3}$) will float.

If mass is measured in g and the volume is in cm^3, then the unit of density will be $g\,cm^{-3}$. Note that $g\,cm^{-3}$ is not an SI unit, but it is nevertheless quite frequently used.

Question

The specimen of granite shown in Figure 3.2 has a mass of 6.20×10^2 g. Calculate the density of the granite in g cm^{-3}.

Answer

The volume of the specimen = 8.4 cm × 5.7 cm × 4.8 cm, so

$$\begin{aligned} \text{density} &= \frac{\text{mass}}{\text{volume}} \\ &= \frac{6.20 \times 10^2 \text{ g}}{8.4 \text{ cm} \times 5.7 \text{ cm} \times 4.8 \text{ cm}} \\ &= 2.6977 \text{ g cm}^{-3} \\ &= 2.7 \text{ g cm}^{-3} \text{ to two significant figures.} \end{aligned}$$

Note that it was not necessary actually to calculate a value for volume before completing the calculation of density. If you had used the value for volume calculated at the beginning of Section 3.4.2, you would have obtained

$$\text{density} = \frac{\text{mass}}{\text{volume}} = \frac{6.20 \times 10^2 \text{ g}}{2.3 \times 10^2 \text{ cm}^3} = 2.7 \text{ g cm}^{-3}$$

but you would have risked introducing rounding errors.

The final worked example in this section converts the units of the density of the granite specimen from g cm^{-3} to kg m^{-3}, using a method which is a combination of the techniques taught throughout Section 3.4. You can convert units of concentration such as mg dm^{-3} to g m^{-3} in a similar way.

Worked example 3.7

Convert $2.7\ \text{g cm}^{-3}$ (the density of the specimen of granite shown in Figures 3.1 and 3.2) to a value in the SI units of kg m^{-3}.

Answer

$1\ \text{kg} = 10^3\ \text{g}$, so $1\ \text{g} = \frac{1}{10^3}\ \text{kg} = 10^{-3}\ \text{kg}$

$1\ \text{m} = 10^2\ \text{cm}$, so $1\ \text{m}^3 = \left(10^2\right)^3\ \text{cm}^3 = 10^6\ \text{cm}^3$ (from Section 3.4.2)

so $1\ \text{cm}^3 = \frac{1}{10^6}\ \text{m}^3 = 10^{-6}\ \text{m}^3$

To convert from g cm^{-3} to kg m^{-3} we need to *multiply* by 10^{-3} (to convert the g to kg) and *divide* by 10^{-6} (to convert the cm^{-3} to m^{-3}).

$$1\ \text{g cm}^{-3} = \frac{10^{-3}}{10^{-6}}\ \text{kg m}^{-3} = 10^{-3-(-6)}\ \text{kg m}^{-3} = 10^3\ \text{kg m}^{-3}$$

Thus $2.7\ \text{g cm}^{-3} = 2.7 \times 10^3\ \text{kg m}^{-3}$.

The specimen of granite has a density of $2.7 \times 10^3\ \text{kg m}^{-3}$.

You may have already known that you need to multiply by 1000 in order to convert from units of $\mathrm{g\,cm^{-3}}$ to units of $\mathrm{kg\,m^{-3}}$, but as was the case with the unit conversions for area and volume, it is better to consider general principles rather than trying to memorize conversion factors.

Question 3.10

The World Health Organization reduced its maximum recommended concentration for arsenic in drinking water from $50\ \mu\mathrm{g\,l^{-1}}$ to $10\ \mu\mathrm{g\,l^{-1}}$ in 1999. Convert $10\ \mu\mathrm{g\,l^{-1}}$ to a value in:

(a) $\mu\mathrm{g\,ml^{-1}}$ Answer

(b) $\mathrm{mg\,dm^{-3}}$ Answer

(c) $\mathrm{g\,m^{-3}}$ Answer

3.5 An introduction to symbols, equations and formulae

To progress further in our exploration of ways of calculating in science, we need to enter the world of symbols, equations and formulae. The word 'algebra' is used to describe the process of using symbols, usually letters, to represent quantities and the relationships between them. Algebra is a powerful shorthand that enables us to describe the relationships between physical quantities briefly and precisely, without having to know their numerical values. Some people consider algebra to be a beautiful thing: others are filled with terror by the very word. This course may not convince you of algebra's beauty, but it should at least illustrate its usefulness and give you an opportunity to learn and practise new techniques or revise old ones.

Chapter 4 is devoted to algebraic techniques such as simplifying, rearranging, and combining equations. The remainder of Chapter 3 simply introduces the language of algebra by looking at a few equations very carefully, and substituting values into them.

The word equation is used for an expression containing an equals sign. The quantities under consideration may be described in words, for example

$$\text{density} = \frac{\text{mass}}{\text{volume}}$$

in which case the equation is known as a 'word equation', or represented by symbols, for example

$$\rho = \frac{m}{V}$$

but the important thing to remember is that what is written on the left-hand side of the '=' sign must *always* be equal to what is written on the right-hand side. Thus, as explained in *Taking care when writing maths* in Section 3.3, you should never use '=' as a shorthand for anything other than 'equals'.

The word formula is used in mathematics to mean a rule expressed in algebraic symbols. Thus $\rho = \frac{m}{V}$ is a formula which tells you that the density ρ of a substance can be obtained by dividing the mass, m, of a sample of the substance by the volume, V, of the sample. Strictly speaking, not all equations are formulae, but the words tend to be used interchangeably.

3.5.1 What do the symbols mean?

Mathematics textbooks teaching algebra frequently contain page after page of equations of the form:

$$x + 3 = 8 \tag{3.1}$$

and

$$y = x + 5 \tag{3.2}$$

In Equation 3.1, x can only have one value, i.e. it is a constant. In this case x has the value 5. In Equation 3.2, x and y are *variables* which can each take an infinite number of values, but y will always be 5 greater than x. The values (of x and y, etc.) which satisfy a particular equation are known as solutions and if you are asked to solve an equation you need to look for solutions.

In both Equation 3.1 and Equation 3.2, x and y represent pure *numbers*. Equations in science are often rather different. Rather than representing pure numbers, the symbols usually represent physical quantities and will therefore have *units* attached.

3.5.2 Which symbols are used

Box 3.4 contains a range of scientific formulae in common use, along with a brief explanation of the meaning of each symbol used. Have a quick at these equations now, but don't worry about their details; you are not expected to learn them or to understand the meanings of the scientific terms introduced. The equations in the boxes will be used as examples throughout the rest of this chapter, and have been numbered for ease of reference.

The symbol chosen to represent something is often the first letter of the quantity in question, e.g. m for mass, t for time and l for length, but it isn't always so simple. Greek letters are also frequently used as symbols e.g. λ (lambda) for wavelength in Equation 3.13 and ρ (rho) for density in Equations 3.9, 3.10 and 3.11. A list of Greek letters and their pronunciation is given in the Table 3.1 and you will soon become familiar with those that are commonly used. In a sense it doesn't matter which symbol you use to represent a quantity, since the symbol is only an arbitrarily chosen label. For instance, Einstein's famous equation (Equation 3.7) is usually written as $E = mc^2$, but the equation could equally well be written using any symbols you wanted to use, e.g. $p = qr^2$, provided you also made it clear that p was used to represent energy, q was used to represent mass and r was used to represent the speed of light. However, the use of conventional symbols, such as E for energy, saves scientists a lot of time in explaining their shorthand. *Maths for Science* follows convention as far as possible in its use of symbols. Sometimes the reason for the choice of symbol will be obvious but unfortunately this is not always the case.

Sometimes a subscript is used alongside a symbol in order to make its meaning more specific, as in v_i, v_f and v_av used in Equation 3.15 to mean initial, final, and average speed, and a_x in Equations 3.16 and 3.17 used to mean acceleration along the x-axis. Note that although a_x, for example, uses two letters, it represents a single physical entity; note also that a_x is *not* the same as ax. The symbol Δ (the Greek upper case delta) is frequently used to represent the change in a quantity, so ΔT in Equation 3.14 means a change in temperature T; again a *single* physical entity is represented by *two* letters.

A few letters have more than one conventional meaning, for example c in Equation 3.7 represents the speed of light, but in Equation 3.14 the same letter represents specific heat capacity. Other letters have two meanings but lower case is conventionally used for one meaning and upper case for the other, for example v for speed and V for volume or t for time and T for temperature. Care needs to be taken, but the intended meaning should be clear from the context.

Unfortunately some Greek letters look rather like everyday English ones; for example ρ (rho), used for density, can look rather like the English lower case p. Some textbooks use lower case p for pressure (this course uses capital P) and Equation 3.11 ($P = \rho g h$) can then appear to have the same quantity on both the left- and right-hand sides of the equals sign, especially when written out by hand. In reality, this formula has *pressure* on the left-hand side and *density* (and other things) on the right-hand side. A similar confusion can arise because the letter l can look like the number 1.

A final possible source of confusion stems from the fact that the same letter may sometimes be used to represent both a physical quantity and a unit of measurement. For example, an object with a mass of 6 kilograms and a length of 2 metres might be described by the relationships $m = 6$ kg, $l = 2$ m, where the letter m is used to represent both mass and the units of length, metres. In all material for this course, and in most other printed text, letters used to represent physical quantities are printed in italics, whereas those used for units are not.

3.5.3 Reading equations

To understand, and thus use, the equations in Box 3.4 you need to be aware of a few rules and conventions. Most of these are extensions of things you have learnt earlier in this course. First:

> When using symbols instead of words or numbers, it is conventional to drop the '×' sign for multiplication.

So in Equation 3.6, *ma* means mass *times* magnitude of acceleration and in Equation 3.11, ρgh means density *times* acceleration due to gravity *times* depth.

> Rules of arithmetic, such as the fact that addition and multiplication are commutative, and the BEDMAS order of operations, apply when using symbols too.

The fact that multiplication is commutative means that equations involving several multiplications can be written in any order. So Equation 3.14 could be (and sometimes is) written as $q = cm\,\Delta T$ instead of $q = mc\,\Delta T$. Addition is also commutative, so Equation 3.16 could be written as $v_x = a_x t + u_x$ instead of $v_x = u_x + a_x t$.

Although the order in which multiplications are written doesn't matter, various conventions are generally applied. Note that in Equation 3.3 ($C = 2\pi r$), the number 2 is written first, then the constant π, then the variable r. This order (numbers, then constants, then variables) is the one that is generally applied. Similarly, $E = mc^2$ (Equation 3.7) could be written as $E = c^2 m$, but it generally isn't! Variables that are raised to a power tend to appear at the end of equations.

BEDMAS tells us that operations within brackets take precedence, i.e. operations inside brackets should be evaluated before those outside the brackets. When working with symbols, this means that an operation applied to a bracket applies to everything within the bracket. So in Equation 3.19, the whole of $\left(\frac{2GM}{R}\right)$ is raised to the power $\frac{1}{2}$. Equation 3.20 uses two sets of brackets (different styles of brackets have been used to avoid confusion). The inner, round brackets ( ) are used to indicate that L should be divided by the whole of $(4\pi F)$ and the outer, square brackets [ ] are used to indicate that the whole of $L/(4\pi F)$ should be raised to the power $\frac{1}{2}$.

There are two further points to note that are linked to the use of brackets.

1 A square root sign and a horizontal line used to indicate division can both be thought of as containing invisible brackets, i.e. the square root sign is taken to apply to everything within the sign and the division applies to everything above the line. So, in Equation 3.10, the square root applies to the whole of $\left(\frac{\mu}{\rho}\right)$, (this means that $\sqrt{\frac{\mu}{\rho}}$ could be written as $\frac{\sqrt{\mu}}{\sqrt{\rho}}$), and in Equation 3.15 the whole of $(v_\mathrm{i} + v_\mathrm{f})$ should be divided by two.

2 Throughout this course, brackets are sometimes used for added clarity even when this is not strictly necessary. In addition, you are encouraged to add your own

brackets whenever you think doing so would make the meaning of an equation clearer.

The 'E' in BEDMAS (see Section 1.4) tells us that exponents take precedence over divisions and multiplications, so in Equation 3.7 ($E = mc^2$) the c must be squared before being multiplied by m. This means that it is *only* the c that is squared, not the m. For clarity you could write this as $E = m\left(c^2\right)$, but it is very important to remember that $mc^2 \neq (mc)^2$, i.e. that $mc^2 \neq m^2c^2$, where the symbol $\neq$ means 'is *not* equal to'.

BEDMAS also reminds us that multiplications should be carried out before additions and subtractions, so in Equation 3.16, a_x and t should be multiplied together before u_x is added.

Finally, note that all of the rules discussed in Chapter 1 for the writing and manipulation of fractions and powers apply when using symbols, in exactly the same way as they do when using numbers. So, Equation 3.17 could be written as $s_x = u_x t + \dfrac{a_x t^2}{2}$ instead of $s_x = u_x t + \frac{1}{2} a_x t^2$; Equation 3.18 could be written as $F_\text{g} = \dfrac{G m_1 m_2}{r^2}$ instead of $F_\text{g} = G\dfrac{m_1 m_2}{r^2}$; and the following two representations of Equation 3.20, although they look very different, are actually identical in meaning:

$$d = \sqrt{\frac{L}{4\pi F}} \qquad\qquad d = [L/(4\pi F)]^{1/2}$$

Question 3.11 Answer

Which *two pairs* of equations for a of those given below are equivalent? You should be able to answer this question by just looking at the equations, but you might like to check your answer by substituting values such as $x = 3$, $y = 4$, $z = 5$.

(i) $a = x(y + z)$

(ii) $a = xy + z$

(iii) $a = (y + z)x$

(iv) $a = x + yz$

(v) $a = z + yx$

Question 3.12 Answer

Two of the equations given below for m are equivalent. Which two? Again, you should attempt this question initially by simply looking at the equations.

(i) $m = \dfrac{bac^2}{d}$

(ii) $m = a\dfrac{b^2c^2}{d}$

(iii) $m = a\dfrac{bc^2}{d}$

(iv) $m = \dfrac{abc^2}{ad}$

(v) $m = \dfrac{b^2a^2c^2}{d}$

3.5.4 Using equations

Substituting values into equations provides a way of checking your understanding of many of the techniques introduced in this chapter, especially the correct reading of equations, the use of scientific notation, and the need to quote answers to an appropriate number of significant figures. It also provides an opportunity for you to extend your understanding of units in calculations and to begin to think about how to choose an appropriate equation to use in answering a particular question. Don't worry about the science in the worked examples in this section; they are given as illustrations of good practice for substituting values into equations.

Worked example 3.8

Use $v_x = u_x + a_x t$ (Equation 3.16) to find the speed reached after 0.45 s by a stone thrown downwards from a cliff with initial speed $1.5\ \mathrm{m\,s^{-1}}$. This situation is illustrated in Figure 3.11. You can assume that the magnitude (size) of the acceleration is $9.81\ \mathrm{m\,s^{-2}}$, where $\mathrm{m\,s^{-2}}$ are the SI units of acceleration.

Answer

Equation 3.16 states that $v_x = u_x + a_x t$, and we are trying to find v_x. The question tells us that

$$u_x = 1.5\ \mathrm{m\,s^{-1}} \quad a_x = 9.81\ \mathrm{m\,s^{-2}} \quad t = 0.45\ \mathrm{s}$$

Thus

$$v_x = \left(1.5\ \mathrm{m\,s^{-1}}\right) + \left(9.81\ \mathrm{m\,s^{-2}} \times 0.45\ \mathrm{s}\right)$$

where the units of a_x are $\mathrm{m\,s^{-2}}$ and the units of t are s, so the units of $a_x t$ are $\mathrm{m\,s^{-2}} \times \mathrm{s}$. Simplifying this gives

$$\mathrm{m\,s^{-2}} \times \mathrm{s} = \frac{\mathrm{m}}{\mathrm{s^2}} \times \mathrm{s} = \frac{\mathrm{m} \times \cancel{\mathrm{s}}}{\mathrm{s} \times \cancel{\mathrm{s}}} = \frac{\mathrm{m}}{\mathrm{s}} = \mathrm{m\,s^{-1}}$$

So

$$\begin{aligned} v_x &= 1.5\ \mathrm{m\,s^{-1}} + 4.4145\ \mathrm{m\,s^{-1}} \\ &= 5.9\ \mathrm{m\,s^{-1}} \text{ to two significant figures,} \end{aligned}$$

i.e. the speed after 0.45 seconds is $5.9\ \mathrm{m\,s^{-1}}$.

Note, from Worked example 3.8, the following points about the handling of units:

1 Calculations have been done in SI units.

2 Units have been included next to values at all times, and the units in the final answers are both consistent with the working *and* what we would expect the units of the final answer to be.

The second point follows from what was said about units in Section 3.1.1; we have input values with units of $\mathrm{m\,s^{-1}}$ for initial speed, units of s for time, and units of $\mathrm{m\,s^{-2}}$ for acceleration, and the units for final speed have *worked out to be* $\mathrm{m\,s^{-1}}$. We have not simply assumed the units for final speed to be $\mathrm{m\,s^{-1}}$, but rather have calculated the units for v_x at the same time as calculating the numerical value. Handling units in this way ensures that the answers are expressed as physical quantities (with units), not just numbers. It also gives an easy way of checking a calculation. If the final units in Worked example 3.8 had come out as $\mathrm{m^2\,s^{-1}}$ you might have realized that, since these are *not* units of speed, you must have made a mistake.

It is good practice to work out the units in this way in *all* your scientific calculations. To enable you to do this, Box 3.5 explains a little more about some of the derived units that you will encounter in this course.

Box 3.5 Derived SI units

Box 2.1 introduced the SI base units, and since then you have encountered the SI units of $m\,s^{-1}$ for speed, $kg\,m^{-3}$ for density and $m\,s^{-2}$ for acceleration. These units are combinations of the base units m, kg and s; other physical quantities have units involving other base units too. Some physical quantities are so commonly used that their units have names and symbols of their own, even though they could be stated as a combination of base units. Several of these derived units are listed in Table 3.2. Note that if you become a sufficiently famous scientist you are likely to end up with a unit named after you! The units in Table 3.2 are named after Sir Isaac Newton, James Prescott Joule, James Watt, Blaise Pascal and Heinrich Hertz respectively.

Physical quantity	Name of unit	Symbol for unit	Base unit equivalent
force, such as weight	newton	N	$kg\,m\,s^{-2}$
energy	joule	J	$kg\,m^{2}\,s^{-2}$
power	watt	W	$kg\,m^{2}\,s^{-3}$
pressure	pascal	Pa	$kg\,m^{-1}\,s^{-2}$
frequency	hertz	Hz	s^{-1}

Table 3.2: Some derived units

Note also that many of the derived units are interlinked:

$$1\ \text{J} = 1\ \text{N} \times 1\ \text{m}$$
$$1\ \text{W} = \frac{1\ \text{J}}{1\ \text{s}}$$
$$1\ \text{Pa} = \frac{1\ \text{N}}{1\ \text{m}^2}$$

The following data may help to illustrate the sizes of the units:

- An eating apple has a weight of about 1 N on Earth;
- An athlete with mass 75 kg, sprinting at 9 m s^{-1}, has an energy of about 3000 J;
- A domestic kettle has a power rating of about 2500 W;
- Atmospheric pressure at sea-level is about 10^5 Pa;
- The human heart beats with a frequency of about 1.3 Hz.

To find the units of v_{esc} in Worked example 3.9, you need to use the fact, from Table 3.2, that $1\ \text{N} = \text{kg}\,\text{m}\,\text{s}^{-2}$. This worked example also provides a reminder of the importance of converting to SI base units before beginning a calculation.

Worked example 3.9

Use $v_{esc} = \left(\frac{2GM}{R}\right)^{1/2}$ (Equation 3.19) to find the escape speed, v_{esc}, needed for an object to escape from the Earth's gravitational attraction. The mass of the Earth, $M = 5.98 \times 10^{24}$ kg, the radius of the Earth, $R = 6.38 \times 10^{3}$ km and $G = 6.673 \times 10^{-11}\ \text{N}\,\text{m}^2\,\text{kg}^{-2}$.

Answer

Converting R to SI base units gives

$$\begin{aligned} R &= 6.38 \times 10^{3}\ \text{km} \\ &= 6.38 \times 10^{3} \times 10^{3}\ \text{m} \\ &= 6.38 \times 10^{6}\ \text{m} \end{aligned}$$

$$M = 5.98 \times 10^{24}\ \text{kg}$$

$$G = 6.673 \times 10^{-11}\ \text{N}\,\text{m}^2\,\text{kg}^{-2}$$

Substituting in Equation 3.19

$$v_{\text{esc}} = \left(\frac{2GM}{R}\right)^{1/2}$$
$$= \left(\frac{2 \times 6.673 \times 10^{-11}\ \text{N}\,\text{m}^2\,\text{kg}^{-2} \times 5.98 \times 10^{24}\ \text{kg}}{6.38 \times 10^{6}\ \text{m}}\right)^{1/2}$$

Rearranging this so that the units on the top of the fraction are all together we get

$$v_{\text{esc}} = \left(\frac{2 \times 6.673 \times 10^{-11} \times 5.98 \times 10^{24}\,\text{N}\,\text{m}^2\,\text{kg}^{-2}\,\text{kg}}{6.38 \times 10^{6}\ \text{m}}\right)^{1/2}$$

Since 1 N = 1 $\text{kg}\,\text{m}\,\text{s}^{-2}$, this can be rewritten as

$$v_{\text{esc}} = \left(\frac{2 \times 6.673 \times 10^{-11} \times 5.98 \times 10^{24}\,\text{kg}\,\text{m}\,\text{s}^{-2}\,\text{m}^2\,\text{kg}^{-2}\,\text{kg}}{6.38 \times 10^{6}\ \text{m}}\right)^{1/2}$$

This can be simplified by cancelling some of the units

$$v_{\text{esc}} = \left(\frac{2 \times 6.673 \times 10^{-11} \times 5.98 \times 10^{24}\ \cancel{\text{kg}}\ \cancel{\text{m}}\ \text{s}^{-2}\ \text{m}^{2}\ \cancel{\text{kg}^{-2}}\ \cancel{\text{kg}}}{6.38 \times 10^{6}\ \cancel{\text{m}}} \right)^{1/2}$$

Calculating the numeric value, and reordering the units, we have

$$v_{\text{esc}} = \left(1.2509 \times 10^{8}\ \text{m}^{2}\,\text{s}^{-2} \right)^{1/2}$$

Taking the square root of both 1.2509×10^{8} and $\text{m}^2\,\text{s}^{-2}$ gives

$$v_{\text{esc}} = 1.12 \times 10^{4}\ \text{m}\,\text{s}^{-1} \text{ to three significant figures.}$$

The escape speed is $1.12 \times 10^{4}\ \text{m}\,\text{s}^{-1}$, with units of $\text{m}\,\text{s}^{-1}$, as expected.

Question 3.13 Answer

In a classic experiment in the USA in 1926, Edgar Transeau calculated the amount of energy stored in the corn plants in a one-acre field in a 100-day growing period to be 1.06×10^8 kJ. This is *NPP* in Equation 3.8. For the same field and the same time period, he found the energy used by the plants in respiration (R) to be 3.23×10^7 kJ. Use Equation 3.8 to find the corresponding value of *GPP*, the total energy captured by the plants.

Question 3.14 Answer

Use Equation 3.13 to find the speed of waves (in m s^{-1}) which have a frequency of 4.83×10^{14} Hz and a wavelength of 621 nm.

The final worked example in this section returns us to the piece of granite introduced at the beginning of the chapter. It is perhaps a somewhat more realistic example than Worked examples 3.8 and 3.9 because the question does not tell us which formula to use.

Worked example 3.10

The rigidity modulus of granite (a measure of the rock's ability to resist deformation) near the surface of the Earth is 3.0×10^{10} Nm^{-2}. Use this value, and the value you found previously for the density of granite to find the speed of S waves travelling through granite.

Answer

Which equation shall we use? When faced by this dilemma it is best to start by thinking carefully about what you already know and what you want to find. On this occasion we're told that the rigidity modulus is 3.0×10^{10} Nm^{-2} and we know (from Worked example 3.7) that the density of granite is 2.7×10^{3} kg m^{-3} (using a value to three significant figures to avoid rounding errors). We need to find a value for S wave speed. So we need an equation which links density, rigidity modulus and S wave speed; Equation 3.10 ($v_{\mathrm{s}} = \sqrt{\frac{\mu}{\rho}}$) from Box 3.4 fits the bill.

Simply finding an equation from a list, all that is possible in this course, is somewhat unlike the situation you are likely to encounter in the real scientific world. Nevertheless, the principle of starting each question by thinking about what you already know and what you want to find is a good one, and on this occasion it makes it straightforward to find an equation to use from Box 3.4.

$$v_s = \sqrt{\frac{\mu}{\rho}}$$

$$\mu = 3.0 \times 10^{10}\ \mathrm{N\,m^{-2}}$$

$$\rho = 2.70 \times 10^{3}\ \mathrm{kg\,m^{-3}}$$

So

$$v_s = \sqrt{\frac{3.0 \times 10^{10}\ \mathrm{N\,m^{-2}}}{2.70 \times 10^{3}\ \mathrm{kg\,m^{-3}}}}$$

Since 1 N = 1 $\mathrm{kg\,m\,s^{-2}}$, this can be rewritten as

$$v_s = \sqrt{\frac{3.0 \times 10^{10}\ \mathrm{kg\,m\,s^{-2}\,m^{-2}}}{2.70 \times 10^{3}\ \mathrm{kg\,m^{-3}}}}$$

This can be simplified by cancelling the kg on top and bottom of the fraction

$$v_s = \sqrt{\frac{3.0 \times 10^{10}\ \cancel{\mathrm{kg}}\,\mathrm{m\,s^{-2}\,m^{-2}}}{2.70 \times 10^{3}\ \cancel{\mathrm{kg}}\,\mathrm{m^{-3}}}}$$

Calculating the numeric value, and combining the m and m^{-2} on the top of the fraction with the m^{-3} on the bottom, we have

$$v_s = \sqrt{1.11 \times 10^7 \text{ m}^2 \text{ s}^{-2}}$$
$$= 3.3 \times 10^3 \text{ m s}^{-1} \text{ to two significant figures}$$

So the S waves travel with a speed of 3.3×10^3 m s^{-1} through granite.

Question 3.15

The Earth has an average radius of 6.38×10^3 km and a mass of 5.97×10^{24} kg. The Moon has a mass of 7.35×10^{22} kg. The distance between the Earth and the Moon is 3.84×10^5 km and $G = 6.673 \times 10^{-11}$ N m^2 kg^{-2}. Use appropriate equations from Box 3.4 to calculate:

(a) the Earth's volume (in m^3); Answer

(b) the magnitude of the gravitational force between the Earth and the Moon (in newtons). Answer

Note: on this occasion you should be able to work out the final units of your answer without expressing newtons in the form of base units. This is further discussed in the answer to the question.

3.6 Learning outcomes for Chapter 3

After completing your work on this chapter you should be able to:

3.1 demonstrate understanding of the terms emboldened in the text;

3.2 perform calculations to an appropriate number of significant figures;

3.3 give answers to calculations in appropriate SI units;

3.4 carry out calculations in scientific notation, both with and without the use of a scientific calculator;

3.5 estimate answers to one significant figure;

3.6 convert between various units for quantities such as area, volume, speed, density and concentration;

3.7 demonstrate understanding of the rules and conventions used in scientific formulae;

3.8 substitute values (numbers and units) into scientific formulae.

4

Algebra

At the end of Chapter 3 we used the equation $v_s = \sqrt{\frac{\mu}{\rho}}$ to calculate the S wave speed, v_s, of seismic waves passing through a rock of density ρ and rigidity modulus μ. But suppose that, instead of knowing ρ and μ and wanting to find v_s, we know v_s and ρ and want to find μ. The best way to proceed is to rearrange $v_s = \sqrt{\frac{\mu}{\rho}}$ to make μ the subject of the equation, where the word 'subject' is used to mean the term written by itself, usually to the left of the equals sign. Rearranging equations is the first topic considered in Chapter 4. The rest of the chapter introduces methods for simplifying equations and ways of combining two or more equations together, and it ends with a look at ways of using algebra to solve problems.

4.1 Rearranging equations

There are many different methods taught for rearranging equations, and if you are happy with a method you have learnt previously it is probably best to stick with this method, provided it gives correct answers to all the questions in this section. However, if you have not found a way of rearranging equations which suits you, you might like to try the method highlighted in the blue-toned boxes throughout this section. This method draws on an analogy between an equation and an old-fashioned set of kitchen scales, and considers the equation to be 'balanced' at the equals sign. The scales will remain balanced if you add a 50 g mass to one side of the scales, or halve the mass on one side, *provided* you do exactly the same thing to the other side. In a similar way, you can do (almost) anything you like to one side of an equation and, provided you do exactly the same thing to the other side, the equation will still be valid. This point is illustrated in Figure 4.1.

The following rule summarizes the discussion above:

> Whatever you do mathematically to one side of an equation you must also do to the other side.

This rule is fundamental when rearranging equations, but it doesn't tell you *what* operation to perform to both sides of an equation in order to rearrange it in the way you want. The highlighted points below should help with this, as will plenty of practice.

Two things are worth noting at the outset:

1 Equations are conventionally written with the subject on the left-hand side of the equals sign. However, when rearranging an equation it can very often be helpful simply to reverse the order.

So if you derive or are given the equation $c = a + b$ you can rewrite it as $a + b = c$; if you derive or are given the equation $ab = c$ you can rewrite it as $c = ab$.

2 Even if you choose the 'wrong' operation, provided you correctly perform that operation to both sides of the equation, the equation will still be valid. Suppose we want to rearrange the equation $c = a + b$ to obtain an expression for a. We could divide by two, as illustrated by Figure 4.1c; this gives

$$\frac{c}{2} = \frac{a + b}{2}$$

This is a perfectly valid equation; it just doesn't help much in our quest for a. The numbered points below give some hints for more helpful ways forward, and each guideline is followed by an illustration of its use.

In the numbered hints the words expression and term are used to describe the parts of an equation. An equation must always include an equals sign, but an expression or term won't. A term may be a single variable (such as v_x or u_x in the equation $v_x = u_x + a_x t$, or a combination of several variables (such as $a_x t$); an expression is usually a combination of variables (such as $a_x t$ or $u_x + a_x t$, but the words are often used interchangeably.

Hint 1

If you want to remove an expression that is *added* to the term you want, *subtract* that expression from both sides of the equation.

To rearrange $a + b = c$ to make a the subject, note that we need to remove the b from the left-hand side of the equation. The b is currently added to a, so we need to subtract b from both sides. This gives

$$a + b - b = c - b$$

or

$$a = c - b \quad \text{(since } b - b = 0\text{)}$$

Hint 2

If you want to remove an expression that is *subtracted* from the term you want, *add* that expression to both sides of the equation.

To rearrange $a - b = c$ to make a the subject, note that we need to remove the b from the left-hand side of the equation. The b is currently subtracted from a, so we need to add b to both sides. This gives

$$a - b + b = c + b$$

or

$$a = c + b \quad (\text{since } -b + b = 0)$$

Hint 3

If the term you want is *multiplied* by another expression, *divide* both sides of the equation by that expression.

To rearrange $ab = c$ to make a the subject, note that we need to remove the b from the left-hand side of the equation. The a is currently multiplied by b, so we need to divide both sides of the equation by b. This gives

$$\frac{ab}{b} = \frac{c}{b}$$

The b in the numerator of the fraction on the left-hand side cancels with the b in the denominator to give

$$a = \frac{c}{b}$$

Hint 4

If the term you want is *divided* by another expression, *multiply* both sides of the equation by that expression.

To rearrange $\frac{a}{b} = c$ to make a the subject, note that we need to remove the b from the left-hand side of the equation. The a is currently divided by b, so we need to multiply both sides of the equation by b. This gives

$$\frac{a \times b}{b} = c \times b$$

The b in the numerator of the fraction on the left-hand side cancels with the b in the denominator to give

$$a = cb$$

Hint 5

If you are trying to make a term the subject of an equation and you currently have an equation for the *square* of that term, take the *square root* of both sides of the equation.

To rearrange $a^2 = b$ to make a the subject, note that the a is currently squared, and take the square root of both sides of the equation. This gives

$$a = \pm\sqrt{b}$$

Note the presence of the $\pm$ sign, indicating that the answer could be either positive or negative, as discussed in Section 1.1.3. In practice, the reality of the problem we are solving sometimes allows us to rule out one of the two values.

Hint 6

If you are trying to make a term the subject of an equation and you currently have an equation for the *square root* of that term, *square* both sides of the equation.

To rearrange $\sqrt{a} = b$ to make a the subject, note that you currently have an equation for the square root of a, and square both sides of the equation. This gives

$$a = b^2$$

Hints 1 to 6 all follow from a general principle:

To 'undo' an operation (e.g. $+$, $-$, $\times$, $\div$, square, square root) you should do the opposite, (i.e. $-$, $+$, $\div$, $\times$, square root, square).

The following worked examples use the principles introduced in the numbered hints above, in the context of equations which are frequently encountered in science. Worked example 4.1 also involves substituting numerical values and units into the equation once it has been rearranged.

Worked example 4.1

As discussed in Box 2.1, mass and weight are not the same. However, the magnitude of the weight, W, of an object at the surface of the Earth and its mass, m, are related by the equation $W = mg$. The magnitude of the acceleration due to gravity, g, can be taken as $9.81\ \text{m s}^{-2}$

A teenager's weight is 649 N. What is his mass?

Answer

We need to start by rearranging $W = mg$ to make m the subject of the equation. It is helpful to start by reversing the order of the equation, i.e. to write it as

$$mg = W$$

To isolate m we need to get rid of g, and m is currently *multiplied* by g so, from Hint 3 we need to *divide* by g. Remember that we must do this to *both sides of the equation*, so we have

$$\frac{mg}{g} = \frac{W}{g}$$

The g in the numerator of the fraction on the left-hand side cancels with the g in the denominator to give

$$m = \frac{W}{g}$$

Substituting values for W and g gives

$$m = \frac{649\ \mathrm{N}}{9.81\ \mathrm{m\,s^{-2}}}$$

Since $1\ \mathrm{N} = 1\ \mathrm{kg\,m\,s^{-2}}$ (from Table 3.2) and

$$\frac{\mathrm{N}}{\mathrm{m\,s^{-2}}} = \frac{\mathrm{kg}\,\cancel{\mathrm{m}}\,\cancel{\mathrm{s^{-2}}}}{\cancel{\mathrm{m}}\,\cancel{\mathrm{s^{-2}}}}$$

we then have

$$m = \frac{649\ \mathrm{kg\,m\,s^{-2}}}{9.81\ \mathrm{m\,s^{-2}}} = 66.2\ \mathrm{kg}$$

So the teenager's mass is 66.2 kg

Worked example 4.2

The time T for one swing of a pendulum is related to its length, L, by the equation

$$T^2 = \frac{4\pi^2 L}{g}$$

where g is the magnitude of the acceleration due to gravity. Write down an equation for T.

Answer

T is currently squared, so from Hint 5, we need to take the square root of both sides of the equation. This gives

$$T = \sqrt{\frac{4\pi^2 L}{g}}$$

Since T is a period of time, its value must be positive, so we only need to write down the positive square root.

Question 4.1

(a) Rearrange $v = f\lambda$ to make f the subject. Answer

(b) Rearrange $E_{\rm tot} = E_{\rm k} + E_{\rm p}$ so that $E_{\rm k}$ is the subject. Answer

(c) Rearrange $\rho = \dfrac{m}{V}$ to obtain an equation for m. Answer

When rearranging more complicated equations, it is often necessary to proceed in several steps. Each step will use the rules already discussed, but many people are perplexed when trying to decide which step to take first. Expertise in this area comes largely with practice, and there are no hard and fast rules (it is often possible to rearrange an equation by several, equally correct, routes). However, the following guidelines may help:

Hint 7

Don't be afraid of using several small steps to rearrange one equation.

Hint 8

Aim to get the new subject into position on the left-hand side as soon as you can. (This will not always be possible straight away.) Simply reversing an equation can sometimes be a helpful initial step.

Hint 9

You can treat an expression within brackets as if it was a single term. This is true whether the brackets are shown explicitly in the original equation or whether you have added them (or imagined them) for clarity. If the quantity required as the subject is itself part of an expression in brackets in the original equation, it is often best to start by making the whole bracketed term the subject of the equation.

Let's look at these guidelines in the context of a series of worked examples, interspersed with questions for you to try for yourself. Note that although 'real' science equations have been used as much as possible in the worked examples and questions, the symbols have not been explained, and you do not need to know the meaning of them. This is to allow you to concentrate, *for the time being only*, on the algebra rather than getting side-tracked into the underlying science.

You may be able to rearrange the equations in the following worked examples in fewer steps than are shown, but if you are in any doubt at all it is best to write down all the intermediate steps in the process.

Worked example 4.3

Rearrange $PV = nRT$ to give an equation for T.

Answer

This example is perhaps more straightforward than it looks, but it is best to proceed in steps.

The first step is to reverse the equation so that the T is on the left-hand side (from Hint 8). This gives

$$nRT = PV$$

We now need to remove the nR by which the T is multiplied. Dividing both sides by nR gives

$$\frac{nRT}{nR} = \frac{PV}{nR}$$

The nR in the numerator of the fraction on the left-hand side cancels with the nR in the denominator to give

$$T = \frac{PV}{nR}$$

Worked example 4.4

Rearrange $\rho = \dfrac{m}{V}$ so that V is the subject.

Answer

The first step is to multiply both sides by V (thus getting V into the right position, as in Hint 8). This gives

$$\rho V = \frac{mV}{V}$$

that is

$$\rho V = m$$

Then dividing both sides by ρ gives

$$\frac{\rho V}{\rho} = \frac{m}{\rho}$$

that is

$$V = \frac{m}{\rho}$$

Worked example 4.5

Rearrange $v_x = u_x + a_x t$ to make u_x the subject.

Answer

This equation can be written as

$$u_x + a_x t = v_x$$

which has u_x on the left-hand side (Hint 8).

We can treat the expression $a_x t$ as a single term (by considering there to be brackets around it, as in Hint 9) and subtract it from both sides of the equation to give

$$u_x + a_x t - a_x t = v_x - a_x t$$

that is

$$u_x = v_x - a_x t$$

Worked example 4.6

Rearrange $h = \frac{1}{2}gt^2$ to give an equation for t.

Answer

We can consider there to be brackets around (t^2) and start by finding an expression for t^2 (Hint 9). The equation can be written as

$$\frac{1}{2}gt^2 = h$$

which has t^2 on the left-hand side (Hint 8). Multiplying both sides by 2 gives

$$2 \times \frac{1}{2}gt^2 = 2h$$

that is

$$gt^2 = 2h$$

Dividing both sides by g gives

$$\frac{gt^2}{g} = \frac{2h}{g}$$

that is

$$t^2 = \frac{2h}{g}$$

Now we can take the square root of both sides to give

$$t = \pm\sqrt{\frac{2h}{g}}$$

Worked example 4.7

Rearrange $v_s = \sqrt{\frac{\mu}{\rho}}$ so that μ is the subject.

Answer

We can consider there to be brackets around $\left(\frac{\mu}{\rho}\right)$ and start by finding an expression for $\left(\frac{\mu}{\rho}\right)$ (Hint 9).

The equation can be written as $\sqrt{\frac{\mu}{\rho}} = v_s$, which has $\frac{\mu}{\rho}$ on the left-hand side (Hint 8).

Squaring both sides gives

$$\frac{\mu}{\rho} = v_s^2$$

Now we can multiply both sides by ρ, to give $\mu = v_s^2 \rho$.

Box 4.1 Interlude: why bother with algebra?

You may have recognized the equation rearranged in Worked example 4.7; it was the one discussed at the beginning of the chapter. Thinking back to the beginning of the chapter reminds us of the purpose of what we are doing. The ability to rearrange equations is useful (arguably the most useful skill developed in this course), but it's not something that you should do just for the sake of doing so, but rather because you want to work something out, and rearranging an equation is the means to this end. Suppose you have been told that S waves pass through rocks of density $\rho = 3.9 \times 10^3\ \mathrm{kg\,m^{-3}}$ with a speed $v_\mathrm{s} = 3.0 \times 10^3\ \mathrm{m\,s^{-1}}$, and you want to find the rigidity modulus μ. The equation in the form $v_\mathrm{s} = \sqrt{\dfrac{\mu}{\rho}}$ is not much use, but the rearranged form immediately tells us that

$$\begin{aligned}\mu &= v_\mathrm{s}^2\,\rho \\ &= \left(3.0 \times 10^3\ \mathrm{m\,s^{-1}}\right)^2 \times \left(3.9 \times 10^3\ \mathrm{kg\,m^{-3}}\right) \\ &= 3.5 \times 10^{10}\ \mathrm{m^2\,s^{-2}\,kg\,m^{-3}} \\ &= 3.5 \times 10^{10}\ \mathrm{kg\,m^{-1}\,s^{-2}}\end{aligned}$$

So the rigidity modulus is $3.5 \times 10^{10}\ \mathrm{kg\,m^{-1}\,s^{-2}}$.

Question 4.2

(a) Rearrange $b = c - d + e$ so that e is the subject. Answer

(b) Rearrange $p = \rho g h$ to give an equation for h. Answer

(c) Rearrange $v_{\mathrm{esc}}^2 = \dfrac{2GM}{R}$ to make R the subject. Answer

(d) Rearrange $E = hf - \phi$ so that ϕ is the subject. Answer

(e) Rearrange $a = \dfrac{bc^2}{d}$ to give an equation for c. Answer

(f) Rearrange $a = \sqrt{\dfrac{b}{c}}$ to make b the subject. Answer

Question 4.3

The mass, m, speed, v, and kinetic energy, E_k, of an object are linked by the equation $E_k = \frac{1}{2}mv^2$.

(a) Rearrange this equation so that v is the subject. Answer

(b) Use your answer to part (a) to estimate (in $m\,s^{-1}$ to one significant figure) the speed needed in order for a tectonic plate of mass 4×10^{21} kg to have a kinetic energy of 2×10^3 J. Answer

(c) Use your answer to part (a) to estimate (in $m\,s^{-1}$ to one significant figure) the speed needed in order for an athlete of mass 70 kg to have the same kinetic energy as the tectonic plate in part (b). Answer

The final group of worked examples in this section involve equations which may appear rather more complex than the previous ones, but they can all be rearranged using the rules and guidelines already introduced. Some, like Worked example 4.8, appear more complex partly because they use symbols that are rather unwieldy. However, these final worked examples are genuinely more complicated too, and are best solved by taking a logical stepwise approach (as the early Arab mathematicians did; see Box 4.2). Rearranging complicated equations is rather like peeling away layers of an onion, systematically removing layer by layer in order to get to the part you want. But that doesn't mean it should end in tears!

Box 4.2 Al-Khwarizmi and al-jabr

The techniques of algebra have developed over a period of several thousand years, but the word ‘algebra’ comes from ‘al-jabr’ in the title of a book written by Mohammed ibn-Musa al-Khwarizmi in about 830. The book, whose title *Hisab al-jabr w’al muqabela*, can be translated as ‘Transposition and reduction’, explained how it was possible to reduce any problem to one of six standard forms using the two processes, al-jabr (transferring terms to eliminate negative quantities) and muqabela (balancing the remaining positive quantities).

Arab mathematicians like al-Khwarizmi did not use symbols in their work, but rather explained everything in words. Nevertheless, their stepwise approach was very similar to the one advocated in this course. Al-Khwarizmi is also remembered for his work on the solution of quadratic equations, discussed later in this chapter.

A little less working is shown in Worked examples 4.8, 4.9 and 4.10 than previously, and hints are not explicitly referred to. This has been done so as to make the working more akin to what you might reasonably write when working through the questions in this course. You are encouraged to show as many steps as necessary in your working, and to use words of explanation wherever they help you.

Worked example 4.8

Rearrange $\Delta G_{\mathrm{m}}^{\ominus} = \Delta H_{\mathrm{m}}^{\ominus} - T\Delta S_{\mathrm{m}}^{\ominus}$ so that $\Delta S_{\mathrm{m}}^{\ominus}$ is the subject.

(*Note:* $\Delta G_{\mathrm{m}}^{\ominus}$, $\Delta H_{\mathrm{m}}^{\ominus}$ and $\Delta S_{\mathrm{m}}^{\ominus}$ each represent a single physical entity.)

Answer

Adding $T\Delta S_{\mathrm{m}}^{\ominus}$ to both sides of the equation gives

$$\Delta G_{\mathrm{m}}^{\ominus} + T\Delta S_{\mathrm{m}}^{\ominus} = \Delta H_{\mathrm{m}}^{\ominus}$$

Subtracting $\Delta G_{\mathrm{m}}^{\ominus}$ from both sides gives

$$T\Delta S_{\mathrm{m}}^{\ominus} = \Delta H_{\mathrm{m}}^{\ominus} - \Delta G_{\mathrm{m}}^{\ominus}$$

Dividing both sides by T gives

$$\Delta S_{\mathrm{m}}^{\ominus} = \frac{\Delta H_{\mathrm{m}}^{\ominus} - \Delta G_{\mathrm{m}}^{\ominus}}{T}$$

Worked example 4.9

Rearrange $s_x = u_x t + \frac{1}{2} a_x t^2$ to make a_x the subject.

Answer

The equation can be written as $u_x t + \frac{1}{2} a_x t^2 = s_x$.

Subtracting $u_x t$ from both sides gives

$$\frac{1}{2} a_x t^2 = s_x - u_x t$$

Multiplying both sides by 2 gives

$$a_x t^2 = 2(s_x - u_x t)$$

Dividing both sides by t^2 gives

$$a_x = \frac{2(s_x - u_x t)}{t^2}$$

Worked example 4.10

Rearrange $F_g = G\dfrac{m_1 m_2}{r^2}$ to give an equation for r.

Answer

Note that $F_g = G\dfrac{m_1 m_2}{r^2}$ can be written as $F_g = \dfrac{Gm_1 m_2}{r^2}$ (see Section 3.5.3).

We can get the r^2 into position on the left-hand side by multiplying both sides by r^2. This gives

$$F_g r^2 = Gm_1 m_2$$

Dividing both sides by F_g gives

$$r^2 = \frac{Gm_1 m_2}{F_g}$$

Taking the square root of both sides gives

$$r = \pm\sqrt{\frac{Gm_1 m_2}{F_g}}$$

Box 4.3 Using algebra in astronomy

The luminosity of a star (the total rate at which it radiates energy into space, in all directions), L, is related to its radius, R, and the temperature (measured in kelvin), T, of its outer layer (called the photosphere) by the equation

$$L = 4\pi R^2 \sigma T^4 \tag{4.1}$$

where σ (the lower case Greek letter sigma) represents a constant known as Stefan's constant, with the value $\sigma = 5.67 \times 10^{-8}\ \mathrm{W\,m^{-2}\,K^{-4}}$.

It is impossible to take direct readings for the luminosity, radius or temperature of distant stars, but indirect measurements can lead to values for photospheric temperature and luminosity. Figure 4.2 is a so-called Hertzsprung–Russell diagram, comparing the photospheric temperatures and luminosity of different stars. Note that different types of stars appear in distinct groupings on the Hertzsprung–Russell diagram.

If we know a star's luminosity and photospheric temperature we can find its radius from Equation 4.1, but first of all we need to rearrange the equation to make R the subject.

Equation 4.1 can be reversed to give

$$4\pi R^2 \sigma T^4 = L$$

Dividing both sides by $4\pi\sigma T^4$ gives

$$R^2 = \frac{L}{4\sigma T^4}$$

(Note that the same results would have been achieved by dividing by 4, π, σ and T^4 separately.)

Taking the square root of both sides gives

$$R = \pm\sqrt{\frac{L}{4\sigma T^4}}$$

Since R is the radius of a star, we are only interested in the positive value.

The star Alcyone (in the Pleiades) has a photospheric temperature of 1.2×10^4 K and a luminosity of 3.2×10^{29} W. So its radius is

$$R = \sqrt{\frac{3.2 \times 10^{29}\ \mathrm{W}}{4 \times 5.67 \times 10^{-8}\ \mathrm{W\,m^{-2}\,K^{-4}} \times (1.2 \times 10^4\ \mathrm{K})^4}}$$

$$R = \sqrt{\frac{3.2 \times 10^{29}\ \cancel{\text{W}}}{4 \times 5.67 \times 10^{-8}\ \cancel{\text{W}}\,\text{m}^{-2}\,\cancel{\text{K}^{-4}}\ (1.2 \times 10^{4})^{4}\ \cancel{\text{K}^{4}}}}$$

$$= \sqrt{2.17 \times 10^{19}\ \text{m}^2}$$

$$= 4.7 \times 10^{9}\ \text{m}$$

The radius of Alcyone is 4.7×10^{9} m.

Notice that in this example, the units of watts cancelled without having to be expressed in SI base units.

Question 4.4

(a) Rearrange $v_x = u_x + a_x t$ so that a_x is the subject. Answer

(b) Rearrange $v_s = \sqrt{\dfrac{\mu}{\rho}}$ to make ρ the subject. Answer

(c) Rearrange $F = \dfrac{L}{4\pi d^2}$ to give an equation for d. Answer

4.2 Simplifying equations

Sometimes it is possible (and helpful) to write an algebraic expression in a different form from the one in which it is originally presented. Whenever possible you should aim to write equations in their simplest form, i.e. to simplify them. For example, you will see in this section that the equation $c = \frac{a}{4b} + \frac{3a}{4b}$ can be simplified to $c = \frac{a}{b}$; the latter form of the equation is rather more useful than the former.

In order to simplify equations it is often necessary to apply the rules for the manipulation of fractions and brackets that were introduced in Chapter 1.

4.2.1 Simplifying algebraic fractions

Algebraic fractions can be multiplied and divided in exactly the same way as numerical fractions, using the methods introduced in Section 1.2.4 and Section 1.2.5. So just as

$$\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15} \quad \text{(multiplying numerators and denominators together)}$$

we can write

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} = \frac{ac}{bd}$$

Similarly, just as

$$\begin{aligned}\frac{2}{3} \div \frac{5}{7} &= \frac{2}{3} \times \frac{7}{5} \quad \text{(turning the } \tfrac{5}{7} \text{ upside down and multiplying)}\\ &= \frac{2 \times 7}{3 \times 5}\\ &= \frac{14}{15}\end{aligned}$$

we can write

$$\begin{aligned}\frac{a}{b} \div \frac{c}{d} &= \frac{a}{b} \times \frac{d}{c} \quad \text{(turning the } \tfrac{c}{d} \text{ upside down and multiplying)}\\ &= \frac{a \times d}{b \times c}\\ &= \frac{ad}{bc}\end{aligned}$$

Worked example 4.11 illustrates a division in which several of the terms cancel.

Worked example 4.11

Simplify $\dfrac{2ab}{c} \div \dfrac{2}{c}$

Answer

Turning the $\dfrac{2}{c}$ upside down and multiplying gives

$$\frac{2ab}{c} \div \frac{2}{c} = \frac{2ab}{c} \times \frac{c}{2}$$

We can cancel the '2's and the 'c's to give

$$\frac{2ab}{c} \div \frac{2}{c} = \frac{\cancel{2}ab}{\cancel{c}} \times \frac{\cancel{c}}{\cancel{2}} = ab$$

The method described in Section 1.2.2 for adding and subtracting numerical fractions can also be extended to algebraic fractions. We need to find a common denominator in a similar way, so, much as we can write

$$\frac{2}{3}+\frac{4}{5}=\frac{2\times 5}{3\times 5}+\frac{4\times 3}{5\times 3}=\frac{10}{15}+\frac{12}{15}=\frac{10+12}{15}=\frac{22}{15}$$

where the common denominator is the product of the denominators of the original fractions, we can also write

$$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{cb}{db}=\frac{ad+cb}{bd}$$

Worked example 4.12

Electrical resistors can be combined together in various ways. You don't need to know or understand the scientific details, but when three resistors of resistance R_1, R_2 and R_3 are combined in a particular way ('in parallel') the effective resistance is given by the term R_{eff} in the equation

$$\frac{1}{R_{\text{eff}}}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3} \tag{4.2}$$

Rearrange Equation 4.2 to make R_{eff} the subject.

Answer

We need to start by expressing the right-hand side of Equation 4.2 as a single fraction. The product of R_1, R_2 and R_3 will be a common denominator, so we can write

$$\begin{aligned}\frac{1}{R_{\text{eff}}} &= \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\\ &= \frac{R_2R_3}{R_1R_2R_3} + \frac{R_1R_3}{R_1R_2R_3} + \frac{R_1R_2}{R_1R_2R_3}\\ &= \frac{R_2R_3 + R_1R_3 + R_1R_2}{R_1R_2R_3}\end{aligned}$$

In order to make R_{eff} the subject of the equation, rather than $\frac{1}{R_{\text{eff}}}$, we could multiply and divide both sides of the equations by a series of expressions. However, it is more straightforward simply to turn the equation upside down, i.e. to take the reciprocal of both sides. This gives

$$R_{\text{eff}} = \frac{R_1R_2R_3}{R_2R_3 + R_1R_3 + R_1R_2}$$

A note of caution when simplifying algebraic expressions

When you simplify an algebraic expression, especially one involving fractions, the answer you arrive at doesn't always look very simple! If you are asked to simplify an expression which is the sum or product of two separate fractions, your answer should normally be a *single* fraction, but an expression like

$$R_{\text{eff}} = \frac{R_1 R_2 R_3}{R_2 R_3 + R_1 R_3 + R_1 R_2}$$

(the answer to Worked example 4.12) may be the simplest you can give. It can be very tempting to 'cancel' terms incorrectly in an attempt to get to the sort of simple fraction which is generally achievable when simplifying numerical fractions, but less likely to be achievable when dealing with symbols.

Question

Express $\dfrac{2c\sqrt{a}}{(a+2)} \times \dfrac{(b+2)}{2c\sqrt{b}}$ as a single fraction of the simplest possible form.

Answer

We can cancel the '2c's to give

$$\frac{\cancel{2c}\sqrt{a}}{(a+2)} \times \frac{(b+2)}{\cancel{2c}\sqrt{b}} = \frac{\sqrt{a}\,(b+2)}{(a+2)\sqrt{b}}$$
$$= \frac{\sqrt{a}\,(b+2)}{\sqrt{b}\,(a+2)}$$

It can be tempting to 'cancel' the square roots and the '+2's too, but this would be incorrect:

$$\frac{\sqrt{a}}{\sqrt{b}} \neq \frac{a}{b} \quad \text{and} \quad \frac{(b+2)}{(a+2)} \neq \frac{b}{a}$$

As discussed in Section 1.2.3, a fraction is unchanged by the multiplication or the division of both its numerator and denominator by the same amount. However, *all other operations will alter its value*.

So $\sqrt{\frac{a}{b}}\,\frac{(b+2)}{(a+2)}$ is as far as it is possible to simplify $\frac{2c\sqrt{a}}{(a+2)} \times \frac{(b+2)}{2c\sqrt{b}}$.

Note however that $\sqrt{\frac{a}{b}}$ is equivalent to $\frac{\sqrt{a}}{\sqrt{b}}$, so $\frac{\sqrt{a}}{\sqrt{b}}\,\frac{(b+2)}{(a+2)}$ can also be written as $\sqrt{\frac{a}{b}}\,\frac{(b+2)}{(a+2)}$.

Question 4.5

Simplify the following expressions, giving each answer as a single fraction.

(a) $\dfrac{\mu_0}{2\pi} \times \dfrac{i_1 i_2}{d}$ Answer

(b) $\dfrac{3a}{2b} \Big/ 2$ Answer

(c) $\dfrac{2b}{c} + \dfrac{3c}{b}$ Answer

(d) $\dfrac{2ab}{c} \div \dfrac{2ac}{b}$ Answer

(e) $\dfrac{1}{f} - \dfrac{1}{f+1}$ Answer

(f) $\dfrac{2b^2}{(b+c)} \div \dfrac{2c^2}{(a+c)}$ Answer

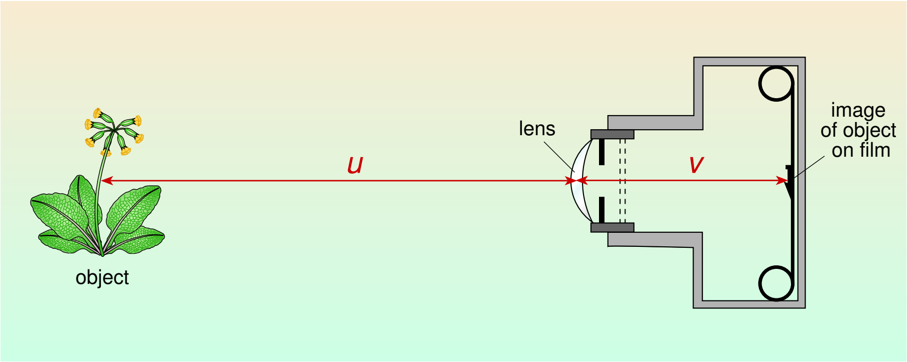


Figure 4.3: The object and image of a simple camera.

Question 4.6 Answer

The distance, u, of an object from a lens (such as the lens in the simple camera illustrated in Figure 4.3) is related to the distance, v, from the lens to the image of the object (on the film) and the lens's focal length, f, by the equation

$$\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$$

Add the fractions $1/u$ and $1/v$ and hence rearrange the equation to give an expression for f.

4.2.2 Using brackets in algebra

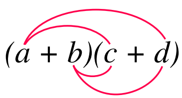
You should be familiar by now with the notion that an operation applied to an expression in a bracket must be applied to *everything* within the bracket, so

$$(2b)^2 = 2^2b^2 = 4b^2$$
$$(a + 2b) - (a + b) = a + 2b - a - b = b$$
$$(a + 2b) - (a - b) = a + 2b - a - (-b) = a + 2b - a + b = 3b$$
$$2(a + 2b) = (2 \times a) + (2 \times 2b) = 2a + 4b$$

and

$$2a(a + 2b) = (2a \times a) + (2a \times 2b) = 2a^2 + 4ab$$

If we need to multiply two bracketed expressions, such as $(a+b)$ and $(c+d)$ together, we need to multiply *each* term in the first bracket by *each* term in the second bracket as indicated by the red lines shown below.



Multiplying the terms in order gives

$$(a+b)(c+d) = \overset{①}{ac} + \overset{②}{ad} + \overset{③}{bc} + \overset{④}{bd}$$

Worked example 4.13

Rewrite the following expressions so that the brackets are removed:

(a) $(x-3)(x+5)$

(b) $(x+y)(x-y)$

(c) $(x+y)^2$

(d) $(x-y)^2$

Answer

(a) $$(x-3)(x+5) = x^2 + 5x - 3x - 15$$
$$= x^2 + 2x - 15$$

(b) $$(x+y)(x-y) = x^2 - xy + yx - y^2$$
$$= x^2 - y^2 \quad \text{since } xy = yx, \text{ so } -xy + yx = 0$$

(c) $(x+y)^2 = (x+y)(x+y)$

$$= x^2 + xy + yx + y^2$$

$$= x^2 + 2xy + y^2$$

(d) $(x-y)^2 = (x-y)(x-y)$

$$= x^2 - xy - yx + y^2$$

$$= x^2 - 2xy + y^2$$

An examination of the answers to parts (b), (c) and (d) of Worked example 4.13 serves as a reminder of the fact that

$$(x+y)^2 \neq x^2 + y^2$$

$$(x-y)^2 \neq x^2 - y^2$$

In other words, remember to watch out for brackets!

Question 4.7

Rewrite the following expressions so that the brackets are removed:

(a) $\frac{1}{2}(v_x + u_x)\,t$ Answer

(b) $\frac{(a - b) - (a - c)}{2}$ Answer

(c) $(k - 2)(k - 3)$ Answer

(d) $(t - 2)^2$ Answer

So far, this section has discussed removing brackets from expressions, but it can very often be useful to do the reverse.

The numbers 6 and 4 are described as factors of 24 and in general, when speaking mathematically, 'factors' are terms which when multiplied together give the original expression. Since, for example,

$$y\,(y + 3) = y^2 + 3y$$

we can say that y and $(y + 3)$ are factors of $y^2 + 3y$

Similarly, since

$$\begin{aligned}(x+3)(x-1) &= x^2 - x + 3x - 3 \\ &= x^2 + 2x - 3\end{aligned}$$

we can say that $(x+3)$ and $(x-1)$ are factors of $x^2 + 2x - 3$.

The verb 'to factorize' means to find the factors of an expression. If you are asked to factorize $y^2 + 3y$ then you should write

$$y^2 + 3y = y(y+3)$$

and if you are asked to factorize $x^2 + 2x - 3$ you should write

$$x^2 + 2x - 3 = (x+3)(x-1)$$

Note, from Worked example 4.13b, that the factors of $x^2 - y^2$ are $(x+y)$ and $(x-y)$, i.e.

$$x^2 - y^2 = (x + y)(x - y) \tag{4.3}$$

The difference of two squared numbers can always be written as the product of their sum and their difference.

Question 4.8

Factorize the following expressions:

(a) $y^2 - y$ Answer

(b) $x^2 - 25$ (*Hint*: you may find it helpful to compare this expression with Equation 4.3, remembering that $5^2 = 25$.) Answer

Factorizing can be useful when rearranging equations, as Worked example 4.14 illustrates.

Worked example 4.14

Rearrange $q = mc\,\Delta T + mL$ so that m is the subject.

Answer

Both terms on the right-hand side of this equation include m, so we can rewrite the equation as

$$q = m\,(c\,\Delta T + L)$$

This can be reversed to give

$$m\,(c\,\Delta T + L) = q$$

Now we divide both sides by $(c\,\Delta T + L)$ to give

$$m = \frac{q}{c\,\Delta T + L}$$

Question 4.9 Answer

Rearrange $E_{\text{tot}} = \frac{1}{2}mv^2 + mg\,\Delta h$ to give an equation for m.

An ability to factorize expressions such as $y^2 + 3y$ and $x^2 + 2x - 3$ can also help us to find the solutions of equations such as $y^2 + 3y = 0$ and $x^2 + 2x - 3 = 0$. Equations of this form are known as 'quadratic equations'.

We know from above that

$$y^2 + 3y = y\,(y + 3) \tag{4.4}$$

So if $y^2 + 3y = 0$, it follows that $y\,(y + 3) = 0$ too. Multiplying by zero gives zero (as discussed in Section 1.1.4). So $y\,(y + 3) = 0$ implies that either $y = 0$ or $y + 3 = 0$.

$y + 3 = 0$ implies that $y = -3$, so the solutions of $y^2 + 3y = 0$ are $y = 0$ and $y = -3$.

We can check that $y = 0$ and $y = -3$ really are solutions of the equation $y^2 + 3y = 0$, by substituting each value for y into the left-hand side of the equation and verifying that it gives 0, as expected.

For $y = 0$, $y^2 + 3y = 0 + 0 = 0$, as expected.

For $y = -3$, $y^2 + 3y = (-3)^2 + \big(3 \times (-3)\big) = 9 + (-9) = 0$, as expected.

It is sensible to check your answers in this way:

> You should check your answers whenever possible.

Worked example 4.15

Use the fact that

$$x^2 + 2x - 3 = (x + 3)(x - 1) \qquad (4.5)$$

to find the solutions of the equation $x^2 + 2x - 3 = 0$.

Answer

If $x^2 + 2x - 3 = 0$ then, from Equation 4.5, $(x + 3)(x - 1) = 0$

Thus $x + 3 = 0$ or $x - 1 = 0$, i.e. $x = -3$ or $x = 1$.

Checking for $x = -3$:

$$x^2 + 2x - 3 = (-3)^2 + 2(-3) - 3 = 9 - 6 - 3 = 0, \text{ as expected.}$$

Checking for $x = 1$:

$$x^2 + 2x - 3 = 1^2 + (2 \times 1) - 3 = 1 + 2 - 3 = 0, \text{ as expected.}$$

So the solutions of the equation $x^2 + 2x - 3 = 0$ are $x = -3$ and $x = 1$.

Using factorization to solve quadratic equations relies on us being able to spot the factors of an expression; this is quite easy for expressions like $y^2 + 3y$ (see Equation 4.4), but if we had not known or been told that $x^2 + 2x - 3 = (x + 3)(x - 1)$ (Equation 4.5), finding the factors of $x^2 + 2x - 3$ would have been largely a matter of trial and error. An ability to find factors in this way can be developed with practice, but it remains somewhat tedious and this method for solving quadratic equations doesn't work at all unless the solutions are whole numbers or simple fractions. Fortunately help is at hand in the form of the 'quadratic equation formula', described in Box 4.4.

Box 4.4 The quadratic equation formula

Al-Khwarizmi and other early Arab mathematicians developed general methods for solving quadratic equations. A quadratic equation of the form

$$ax^2 + bx + c = 0$$

will have solutions given by the quadratic equation formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 > 4ac$ (i.e. b^2 is greater than $4ac$) then $b^2 - 4ac$ will be positive, and the formula will lead to two distinct solutions.

If $b^2 = 4ac$ then $b^2 - 4ac = 0$, so the two solutions will be identical ($x = -b/(2a)$).

If $b^2 < 4ac$ (i.e. b^2 is less than $4ac$) then $b^2 - 4ac$ will be negative. This means that the solutions will include the square root of a negative number. and hence will involve 'imaginary numbers'. Such numbers were mentioned in Chapter 1, but will not be considered further in *Maths for Science*.

Worked example 4.16 demonstrates the use of the quadratic equation formula in solving the equation that was solved by factorization in Worked example 4.15.

Worked example 4.16

Use the quadratic equation formula to find the solutions of the equation $x^2 + 2x - 3 = 0$.

Answer

Comparison of

$$x^2 + 2x - 3 = 0$$

and

$$ax^2 + bx + c = 0$$

shows that $a = 1$, $b = 2$ and $c = -3$ on this occasion, so the solutions are

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - \left(4 \times 1 \times (-3)\right)}}{2 \times 1} \\ &= \frac{-2 \pm \sqrt{4 - (-12)}}{2} \\ &= \frac{-2 \pm \sqrt{16}}{2} \\ &= \frac{-2 \pm 4}{2} \end{aligned}$$

So $x = \dfrac{-2 + 4}{2} = 1$

or $x = \dfrac{-2 - 4}{2} = -3$

The solutions can be checked in exactly the same way as in Worked example 4.15.

Once again, we have found that the solutions of the equation $x^2 + 2x - 3 = 0$ are $x = -3$ and $x = 1$.

Question 4.10

(a) Use your answer to Question 4.7 (c) to solve $k^2 - 5k + 6 = 0$ by factorization. Answer

(b) Use your answer to Question 4.7 (d) to solve $t^2 - 4t + 4 = 0$ by factorization. Answer

(c) Use the quadratic equation formula to check your answer to part (a). Answer

(d) Use the quadratic equation formula to check your answer to part (b). Answer

4.3 Combining equations

Consider the equation $E = hf$. This equation, first proposed by Einstein, links the energy, E, of light to its frequency, f (h is a constant known as Planck's constant). Suppose that you know h and are trying to find E, but that you don't know f. Instead you know the values of c (speed of light) and λ (wavelength) in a second equation, $c = f\lambda$. It would be possible to calculate a value for f from the second equation and then substitute this value in the first equation so as to find E. However, this approach runs the risk of numerical slips and rounding errors. It is more useful to do the substitution *algebraically*, in the way shown in the following example.

Worked example 4.17

Combine the following two equations to find an equation for E not involving f:

$$E = hf \tag{4.6}$$

$$c = f\lambda \tag{4.7}$$

Answer

Rearranging Equation 4.7 gives

$$f = \frac{c}{\lambda}$$

Substituting this expression for f into Equation 4.6 gives

$$E = h \times \frac{c}{\lambda} = \frac{hc}{\lambda}$$

This mathematical technique, sometimes referred to as elimination (because a variable, f on this occasion, is being eliminated), can be used in many situations, as illustrated in the worked examples throughout this section.

Worked example 4.18

Combine $F_g = G\frac{Mm}{r^2}$ and $F_g = mg$ to give an equation for r not involving F_g.

Answer

Since both equations are already given with F_g (the variable we are trying to eliminate) as the subject, we can simply set the two equations for F_g equal to each other:

$$mg = G\frac{Mm}{r^2}$$

We now need to rearrange to give an equation for r. First note that there is an m on both sides of the equation, so we can divide both sides of the equation by m to give

$$g = G\frac{M}{r^2}$$

Multiplying both sides by r^2 gives

$$gr^2 = GM$$

Dividing both sides by g gives

$$r^2 = \frac{GM}{g}$$

Taking the square root of both sides gives

$$r = \pm\sqrt{\frac{GM}{g}}$$

Question 4.11

(a) Combine $E_k = \frac{1}{2}mv^2$ and $p = mv$ to give an equation for E_k not involving v. Answer

(b) Combine $E = \frac{1}{2}mv^2$ and $E = mg\,\Delta h$ to give an equation for v not involving E. Answer

(c) Combine $E_k = hf - \phi$ and $c = f\lambda$ to give an equation for ϕ not involving f. Answer

Two (or more) different equations containing the same two (or more) unknown quantities are called 'simultaneous equations' if the equations must be satisfied (hold true) simultaneously. It is usually possible to solve two simultaneous equations by using one equation to eliminate one of the unknown quantities from the second equation, in an extension of the method discussed above. This is illustrated in Worked example 4.19.

Worked example 4.19

Find values of x and y which satisfy both of the equations given below:

$$x + y = 7 \tag{4.8}$$

$$2x - y = 2 \tag{4.9}$$

Answer

If we rewrite Equation 4.8 to give an equation for y in terms of x, then we can insert this result into Equation 4.9 to give an equation for x alone.

Subtracting x from both sides of Equation 4.8 gives

$$y = 7 - x \tag{4.10}$$

Substituting for y in Equation 4.9 then gives

$$
\begin{aligned}
& 2x - (7 - x) = 2 \\
\text{i.e. } & 2x - 7 + x = 2 \\
\text{or } & 3x - 7 = 2
\end{aligned}
$$

Adding 7 to both sides gives

$$3x = 9, \text{ i.e. } x = 3$$

Substitution of $x = 3$ into Equation 4.10 shows that

$$y = 7 - x = 7 - 3 = 4$$

So the solution (i.e. the values for x and y for which both of the equations hold true) is $x = 3$ and $y = 4$. We can check this by substituting the values for x and y into the left-hand side of Equations 4.8 and 4.9.

Equation 4.8 gives $x + y = 3 + 4 = 7$, as expected.
Equation 4.9 gives $2x - y = (2 \times 3) - 4 = 6 - 4 = 2$, as expected.

We could have arrived at the same result by using Equations 4.8 and 4.9 in a different order, but there is only one correct answer.

Worked example 4.19 shows that in order to find two unknown quantities, two different equations relating them are required. This is always true and by extension:

> When you combine equations so as to find unknown quantities, it is always necessary to have at least as many equations as there are unknown quantities.

Worked example 4.20 shows how four equations can be combined together in a case where there are four unknown quantities (we are trying to find the total surface area, S, but the mass, m, and volume, V, of a single particle and the number of particles, n, are unknown too and so must be eliminated). This worked example concerns the use of metal particles as catalysts in the chemical industry (see Box 4.5).

Box 4.5 Chemical catalysts

A catalyst is a substance which makes a chemical reaction proceed more rapidly. The catalyst itself does not undergo permanent chemical change, and it can be recovered when the chemical reaction is completed. Metal particles can be used as catalysts. A large number of small particles will have a greater surface area than a small number of larger particles, and the total surface area, S, of the particles is of critical importance to the speed of the reaction. In a typical industrial chemical reactor, S can be approximately 5000 km^2; roughly a third the area of Yorkshire!

Worked example 4.20

The total surface area, S, of n metal particles of average radius r is given by the equation

$$S = 4\pi nr^2 \tag{4.11}$$

The number of particles n is linked to the mass of one particle, m and the total mass of metal, M by the equation

$$n = \frac{M}{m} \tag{4.12}$$

The mass m of one particle is linked to its volume V and the density of the metal ρ by the equation

$$\rho = \frac{m}{V} \tag{4.13}$$

The volume V of a particle is given by

$$V = \frac{4}{3}\pi r^3 \tag{4.14}$$

where r is the radius.

Find an equation for S in terms of M, ρ and r only.

Answer

Reversing Equation 4.13 gives

$$\frac{m}{V} = \rho$$

Multiplying both sides by V gives

$$m = V\rho$$

Substituting for V from Equation 4.14 gives

$$m = \frac{4}{3}\pi r^3 \rho$$

Substituting this expression for m into Equation 4.12 gives

$$\begin{aligned} n &= \frac{M}{m} \\ &= \frac{M}{\frac{4}{3}\pi r^3 \rho} \\ &= \frac{3M}{4\pi r^3 \rho} \end{aligned}$$

Substituting this expression for n into Equation 4.11 gives

$$\begin{aligned} S &= 4\pi n r^2 \\ &= \cancel{4\pi} \times \frac{3M}{\cancel{4\pi} r^{\cancel{3}} \rho} \times \cancel{r^2} \\ &= \frac{3M}{r\rho} \end{aligned}$$

4.4 Putting algebra to work

So far, Chapter 4 has been concerned almost exclusively with symbols. Equations have been given to you and you have been told to manipulate them in a particular way. In the real scientific world, you are likely to need to:

1 Choose the correct equation(s) to use or derive equation(s) for yourself.

2 Combine, rearrange and simplify the equation(s) using the skills introduced in the earlier sections of this chapter.

3 Substitute numerical values, taking care over things like significant figures, scientific notation and units, as you did in Chapter 3.

4 Check that the answer is reasonable.

The final section of this chapter considers these points, combining skills from Chapters 3 and 4, but it starts with a more light-hearted look at the uses of algebra.

4.4.1 Algebra is fun!

Try this:

- Think of a number.
- Double it.
- Add four.
- Halve your answer.
- Subtract 1.

If you have arrived at an answer of 4, I can tell you that the number you first thought of was 3; if your answer is 6, the number you first thought of was 5, if your answer is 11, the number you first thought of was 10, and so on.

Magic? No, a demonstration of the power of algebra! We could perform exactly the same operations for *any* number; let's represent the number by the symbol N. Then we have

- Think of a number. N
- Double it. $2N$

- Add four. $2N + 4$
- Halve your answer. $\frac{1}{2}(2N + 4) = N + 2$
- Subtract 1. $(N + 2) - 1 = N + 1$

So the final answer will always be one more than the number you first thought of.

Here's another one for you to try:

- Think of a number.
- Add 5.
- Double the result.
- Subtract 2.
- Divide by 2.
- Take away the number you first thought of.

Whatever number you first thought of, the answer will always be four.

Question 4.12 Answer

Use a symbol of your choice to represent the number in the 'think of a number' example immediately before this question and thus show that the answer will be four, whatever number you choose at the beginning.

You may wonder why a course entitled *Maths for Science* has suddenly started discussing number tricks. There is a serious point to this, namely to illustrate how you can get from an initial problem to a solution by using algebra. Worked example 4.21 illustrates another use of algebra.

Worked example 4.21

Chris and Jo share a birthday (but are different ages). On their birthday this year Chris will be five times older than Jo. Their combined age on their birthday last year was 58. How old was Chris when Jo was born?

Answer

Let C represent Chris's age in years on her birthday this year and J represent Jo's age in years on her birthday this year.

Since Chris will be five times older than Jo we can say

$$C = 5J \tag{4.15}$$

Last year Chris's age was $(C-1)$ and Jo's age was $(J-1)$, so we can say

$$(C-1)+(J-1)=58$$
$$\text{i.e. } C+J-2=58$$
$$C+J=60 \qquad (4.16)$$

Substituting for C from Equation 4.15 in Equation 4.16 gives

$$5J+J=60$$
$$\text{i.e. } 6J=60$$
$$J=10$$

Thus, from Equation 4.15, $C=5\times 10=50$.

Thus Chris will be 50 this year and Jo will be 10. But this wasn't the question that was asked! When Jo was born, Chris was 50 – 10, i.e. 40 years old.

You may remember questions like Worked example 4.21 from your school days. Problems like this can seem intimidating, but they are relatively easy to solve once you have found the equations that describe the problem. Many people struggle with this first step — they can't find the equations to use. Look at Worked example 4.21 carefully; all that has been done in order to derive Equation 4.15 and Equation 4.16 has been to study carefully the information given in the question, and to write it down in terms of symbols. So 'On their birthday this year Chris will be five times older than Jo' has become $C = 5J$. In solving problems, it is almost always helpful to start by writing down what you already know. Drawing a diagram to illustrate the situation can help too; you may find this helpful in Question 4.13.

Question 4.13 Answer

Tracey is 15 cm taller than Helen, and when Helen stands on Tracey's shoulder she can just see over a fence 3 m tall. Assume that it is 25 cm from Tracey's shoulder to the top of her head and 10 cm from Helen's eyes to the top of her head. How tall is Helen?

4.4.2 Using algebra to solve scientific problems

In much the same way as people struggle when trying to derive equations for use in problems like Worked example 4.21, they often have difficulty deciding which formulae to use from those given in a book or on a formula sheet. Again, it can be helpful to draw a diagram and it is *always* helpful to start by writing down what you know and what you're trying to find. This often helps you to decide how to proceed.

Worked example 4.22 discusses the choice of appropriate formulae for use in answering a particular question. It also works through the other steps you are likely to follow when using algebra to solve scientific problems.

Worked example 4.22

A silver sphere (density 10.49 g cm^{-3}) has a radius of 2.5 mm. What is its mass? Use formulae given in Box 3.4.

Which equations shall we use?

We know density (ρ) and radius (r) and are trying to find mass (m), so we need an equation to link these three variables. Equation 3.9, $\rho = \dfrac{m}{V}$, links density and mass, but it also includes *volume* which isn't either given or required by the question. Fortunately help is at hand in the form of Equation 3.5, $V = \frac{4}{3}\pi r^3$ which gives the volume V of a sphere of radius r. We should be able to substitute for V from Equation 3.5 into Equation 3.9. This will give an equation involving only ρ, r and m, as required, and we can then rearrange it to make m the subject.

Combining and rearranging equations

Substituting for V from Equation 3.5 into Equation 3.9 gives

$$\rho = \frac{m}{\frac{4}{3}\pi r^3}$$

Multiplying top and botom of the fraction by 3 gives

$$\rho = \frac{3m}{4\pi r^3}$$

Reversing this so that m is on the left-hand side gives

$$\frac{3m}{4\pi r^3} = \rho$$

Multiplying both sides by $4\pi r^3$ gives

$$3m = 4\pi r^3 \rho$$

Dividing both sides by 3 gives

$$m = \frac{4}{3}\pi r^3 \rho$$

Substituting numerical values

Note that we have used symbols for as long as possible in this question, so as to avoid numerical slips and rounding errors. However, we are now almost ready to substitute the values given for r and ρ. First we need to convert the values given into consistent (preferably SI) units:

$$r = 2.5 \text{ mm} = 2.5 \times 10^{-3} \text{ m}$$

$\rho = 10.49 \text{ g cm}^{-3} = 10.49 \times 10^{3} \text{ kg kg}^{-3}$ ($1.049 \times 10^{4} \text{ kg kg}^{-3}$ in scientific notation), converting from g cm^{-3} to kg m^{-3} in the way described in Section 3.4.4. Then

$$\begin{aligned} m &= \frac{4}{3}\pi r^3 \rho \\ &= \frac{4}{3}\pi \left(2.5 \times 10^{-3} \text{ m}\right)^3 \times 1.049 \times 10^{4} \text{ kg kg}^{-3} \\ &= 6.9 \times 10^{-4} \cancel{\text{m}^3} \text{ kg} \cancel{\text{m}^{-3}} \\ &= 6.9 \times 10^{-4} \text{ kg} \end{aligned}$$

Is the answer reasonable?

It is always worth spending a few minutes checking whether the answer you have arrived at is reasonable. There are three simple ways of doing this (it is not normally necessary to use all three methods to check one answer):

1 We can check the units of the answer. We have given units next to all the numerical values in the calculation, and the units on the right-hand side of the equation have worked out to be kilograms, as we would expect for mass.

If we had made a mistake in transposing the formula for mass, and had written it as $m = \frac{4}{3}\pi r^2 \rho$ by mistake, then the units on the right-hand side would have been $\mathrm{m}^2 \times \mathrm{kg\,m}^{-3} = \mathrm{kgm}^{-1}$. These are not units expected for mass by itself, so we would have been alerted to the fact that something was wrong.

Checking units in this way provides a good way of checking that you have written down or derived an equation correctly; the units on the left-hand side of an equation should always be equal to the units on the right-hand side. You can use this method for checking an equation even if you are not substituting numerical values into it.

2 We can estimate the value (in the way described in Section 3.3), and compare it with the answer found on a calculator. In this case

$$
\begin{aligned}
m &\approx \frac{4}{3} \times 3\left(3 \times 10^{-3}\ \mathrm{m}\right)^3 \times 1 \times 10^4\ \mathrm{kg\,m}^{-3} \\
&\approx \frac{4}{\cancel{3}} \times \cancel{3} \times 3^3 \times 10^{-9}\ \cancel{\mathrm{m}^3} \times 1 \times 10^4\ \mathrm{kg}\,\cancel{\mathrm{m}^{-3}} \\
&\approx 4 \times 27 \times 10^{-9+4}\ \mathrm{kg} \\
&\approx 100 \times 10^{-5}\ \mathrm{kg} \\
&\approx 1 \times 10^{-3}\ \mathrm{kg}
\end{aligned}
$$

This is the same order of magnitude as the calculated value, so the calculated value seems reasonable.

3 We can look at the answer and see if it is what common sense might lead us to expect. Obviously this method only works when you are doing a calculation concerning physical objects with which you are familiar, but it gives a sensible check for worked examples like the one we are considering. It seems reasonable that a silver sphere with a diameter of 0.5 cm might have a mass of something less than a gram. If you'd arrived at an answer of 1.1×10^2 kg (by forgetting to cube the value given for r) you might have thought that this mass (equivalent to more than 100 bags of sugar!) was rather large for such a small sphere.

Note that checking doesn't usually tell you that your answer is absolutely correct — none of the methods described above would have spotted small arithmetic slips — but it does frequently alert you if the answer is wrong.

Tips for using algebra to solve scientific problems

1. Start by writing down what you know and what you're trying to find, and use this information to find appropriate equations to use.
2. Combine, rearrange and simplify the equations, using symbols for as long as possible so as to avoid numerical slips and rounding errors.
3. When you substitute numerical values, take care with units, scientific notation and significant figures.
4. Check that your final answer is reasonable, by asking yourself the following questions:
   (a) Are the units what you would expect?
   (b) Is the answer similar to the one you have obtained by estimating?
   (c) Is the answer about what you would expect from common sense?

Worked example 4.23 shows the use of these tips in solving a different problem, concerning the conservation of energy. This worked example uses formulae introduced in Box 4.6; you may also find these formulae useful when answering Question 4.14.

Box 4.6 The conservation of energy

Energy can never be destroyed, but it is frequently converted from one form to another. As a child climbs the steps of a slide, he or she gains in gravitational potential energy; as he or she slides down the slide this energy is converted into kinetic (movement) energy. As a kettle boils, the electrical energy increases the energy of the water molecules and so raises the temperature of the water. In both cases some energy is 'lost' to other forms (such as heat to the surroundings and sound) but very often you can assume that all of the energy initially in one form is converted to just one other form, and so equate formulae (such as those given below) for different forms of energy. All forms of energy should be quoted using the SI unit of energy which is the joule (J), where $1\ \mathrm{J} = 1\ \mathrm{kg\,m^2\,s^{-2}}$.

The kinetic energy (energy of motion), E_k, of an object with a mass m moving at speed v is given by

$$E_\mathrm{k} = \tfrac{1}{2}mv^2 \qquad (4.17)$$

The gravitational potential energy, E_g, of an object of mass m at a height Δh above a reference level is given by

$$E_\mathrm{g} = mg\,\Delta h \qquad (4.18)$$

where g is the acceleration due to gravity.

The energy, q, needed to raise the temperature of a mass m of a substance of specific heat capacity c by a temperature ΔT is given by

$$q = mc\,\Delta T \tag{4.19}$$

Worked example 4.23

A lump of putty is dropped from a height of 4.8 m. The putty's gravitational potential energy is all converted into kinetic energy as it falls. If, on impact, all of this kinetic energy is used to raise the temperature of the putty, by how much does the temperature of the putty rise? Assume that the specific heat capacity of the putty is $5.0 \times 10^2\ \mathrm{J\,kg^{-1}\,K^{-1}}$ and that the acceleration due to gravity is $9.81\ \mathrm{m\,s^{-2}}$.

Which equations shall we use?

It is tempting to involve Equation 4.17, as the question talks about the putty's kinetic energy, but closer inspection of the question reveals that we can assume that all the gravitational potential energy becomes kinetic energy as the putty falls, and that all the kinetic energy is transferred to heat energy in the putty on impact. So we can say that all the gravitational potential energy is transferred to heat energy; we simply need to set Equations 4.18 and 4.19 equal to each other. We have not been told the mass of the putty, but since the term m appears in both Equation 4.18 and Equation 4.19 we will be able to cancel this term, which will leave us with an equation linking g, Δh, c and ΔT. We know g, Δh and c and are trying to find ΔT.

Combining and rearranging equations

Since we can assume that all the gravitational potential energy, E_g, is transferred to heat energy, q, we can set Equation 4.18 and Equation 4.19 equal to each other.

$$mc\,\Delta T = mg\,\Delta h$$

There is an m on both sides, so we can divide by m to give

$$c\,\Delta T = g\,\Delta h$$

Dividing both sides by c gives

$$\Delta T = \frac{g\,\Delta h}{c}$$

Substituting numerical values

$$g = 9.81\ \mathrm{m\,s^{-2}}$$
$$h = 4.8\ \mathrm{m}$$
$$c = 5.0 \times 10^2\ \mathrm{J\,kg^{-1}\,K^{-1}}$$

so

$$\begin{aligned}\Delta T &= \frac{g\,\Delta h}{c}\\ &= \frac{9.81\ \mathrm{m\,s^{-2}} \times 4.8\ \mathrm{m}}{5.0 \times 10^2\ \mathrm{J\,kg^{-1}\,K^{-1}}}\\ &= \frac{9.81 \times 4.8\ \cancel{\mathrm{m}}\,\cancel{\mathrm{s^{-2}}} \times \cancel{\mathrm{m}}}{5.0 \times 10^2\ \cancel{\mathrm{kg}}\,\cancel{\mathrm{m^2}}\,\cancel{\mathrm{s^{-2}}}\,\cancel{\mathrm{kg^{-1}}}\,\mathrm{K^{-1}}}\\ &= 0.094\ \mathrm{K}\ \text{to two significant figures.}\end{aligned}$$

Is the answer reasonable?

In a real question you probably wouldn't use all the checks described in the blue-toned box after Worked example 4.22, but the answer seems about the size you might expect (you wouldn't expect a big temperature rise) and the units have worked out to be kelvin, as expected for a change in temperature.

Alternatively we can estimate the answer to be

$$\Delta T \approx \frac{10\ \mathrm{m\,s^{-2}} \times 5\ \mathrm{m}}{5 \times 10^2\ \mathrm{J\,kg^{-1}\,K^{-1}}} \approx 10^{-1}\ \mathrm{K}$$

This is the same order of magnitude as the calculated value, so the calculated value seems reasonable.

Question 4.14 Answer

A child climbs to the top of a 1.8 m slide and then slides to the ground. Assuming that all of her gravitational potential energy is converted into kinetic energy, find her speed as she reaches the ground. Take $g = 9.81\ \mathrm{m\,s^{-2}}$ and use appropriate formulae from Box 4.6.

In Worked example 4.24, the final worked example in Chapter 4, we return to a discussion of seismic waves travelling through the Earth's crust (as introduced in Box 3.1). In this example there are three unknown quantities (the distance, d, from the earthquake, the time, t_{P}, taken for P waves to reach the seismometer and the time, t_{S}, taken for S waves to reach the seismometer) so we need to combine three equations to find any of the unknown quantities. You will not be expected to combine more than two equations together in any questions associated with this course, but Worked example 4.24 has been included because it summarizes much of what has been discussed in Chapter 4, and also because it illustrates the usefulness of algebra in science.

Box 4.7 Locating an earthquake

Figure 4.4 shows a seismogram recorded at the British Geological Survey in Edinburgh on 12 September 1988. It is possible to see the points at which P waves and S waves first reached the seismometer. We can assume that these seismic waves originated in an earthquake somewhere. But where was the earthquake and when did it occur? (although the recording was made at 2.23 p.m., it does not tell us the time at which the earthquake occurred, since the waves will have taken some time to reach the seismometer from the point of origin or focus of the earthquake).

Figure 4.4 shows that the P waves reached the seismometer 20 seconds before the S waves.

We assume that the P waves travelled with an average speed, $v_p = 5.6\ \text{km s}^{-1}$ and that the S waves travelled with an average speed $v_s = 3.4\ \text{km s}^{-1}$ (these values are typical for the rocks of the Earth's crust, through which the waves will have been travelling).

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

$$\text{so} \quad v_p = \frac{d}{t_p} \qquad (4.20)$$

$$\text{and} \quad v_s = \frac{d}{t_s} \qquad (4.21)$$

where d is the distance from the earthquake, t_p is the time taken for P waves to travel to the seismometer and t_s is the time taken for S waves to travel to the seismometer.

Worked example 4.24

Use the information given in Box 4.7 to find the distance from Edinburgh to the focus of the earthquake recorded on the seismogram shown in Figure 4.4.

Which equations shall we use?

We know that $v_p = \dfrac{d}{t_p}$ (Equation 4.20) and $v_s = \dfrac{d}{t_s}$ (Equation 4.21), where $v_p = 5.6\ \text{km}\,\text{s}^{-1}$ and $v_s = 3.4\ \text{km}\,\text{s}^{-1}$, but d, t_p and t_s are all unknown, so we need another equation.

Although we don't know the travel time of the two types of wave, we know that the difference in the arrival time of the two waves is 20 seconds, so we can write

$$t = t_s - t_p \tag{4.22}$$

where $t = 20$ s.

Equations 4.20, 4.21 and 4.22 give us three equations containing the three unknowns d, t_p and t_s and we need to combine and rearrange them to give an expression for d.

Combining and rearranging equations

Multiplying both sides of Equation 4.20 by t_p gives

$$t_p v_p = d$$

Dividing both sides by v_p gives

$$t_p = \frac{d}{v_p}$$

Similarly, from Equation 4.21,

$$t_s = \frac{d}{v_s}$$

Substituting for t_s and t_p in Equation 4.22 gives

$$\begin{aligned} t &= t_s - t_p \\ &= \frac{d}{v_s} - \frac{d}{v_p} \\ &= d\left(\frac{1}{v_s} - \frac{1}{v_p}\right) \end{aligned}$$

Combining the fractions by making $v_s v_p$ a common denominator (Section 4.2.1) gives

$$t = d\,\frac{(v_p - v_s)}{v_s v_p}$$

Reversing the equation so that d is on the left-hand side gives

$$d\,\frac{(v_p - v_s)}{v_s v_p} = t$$

Multiplying both sides by $v_s v_p$ gives

$$d\,(v_p - v_s) = t\,v_s v_p$$

Dividing both sides by $(v_p - v_s)$ gives

$$d = \frac{t\,v_s v_p}{v_p - v_s}$$

Substituting numerical values

Substituting $t = 20$ s, $v_p = 5.6\ \text{km s}^{-1}$ and $v_s = 3.4\ \text{km s}^{-1}$ gives

$$\begin{aligned} d &= \frac{20\ \text{s} \times 3.4\ \text{km s}^{-1} \times 5.6\ \text{km s}^{-1}}{(5.6\ \text{km s}^{-1} - 3.4\ \text{km s}^{-1})} \\ &= \frac{20\ \text{s} \times 3.4\ \text{km s}^{-1} \times 5.6\ \text{km s}^{-1}}{2.2\ \text{km s}^{-1}} \\ &= 1.7 \times 10^2\ \text{km to two significant figures} \end{aligned}$$

The units work out to be kilometres since $\frac{\cancel{\text{s}} \times \text{km}\,\cancel{\text{s}^{-1}} \times \cancel{\text{km}}\,\cancel{\text{s}^{-1}}}{\cancel{\text{km}}\,\cancel{\text{s}^{-1}}} = \text{km}$

Is the answer reasonable?

The units have worked out to be kilometres as expected for a distance. If we had converted the speeds to values in ms^{-1}, we would have obtained a value for d in metres ($d = 1.7 \times 10^5$ m).

In this case it is easy to check that the answer is reasonable; many members of the public reported a small earthquake on that day in Ambleside in Cumbria. Ambleside is 170.5 km from Edinburgh!

In general, to use this method to uniquely identify the location of an earthquake you need to repeat the exercise using data received at other seismometers elsewhere on the Earth's surface.

4.5 Learning outcomes for Chapter 4

After completing your work on this chapter you should be able to:

4.1 demonstrate understanding of the terms emboldened in the text;

4.2 rearrange an algebraic equation to make a different variable the subject;

4.3 simplify an algebraic expression;

4.4 add, subtract, multiply and divide algebraic fractions;

4.5 re-write an algebraic expression so that the brackets are removed;

4.6 factorize a simple algebraic expression;

4.7 eliminate one or more variables so as to combine equations together;

4.8 check the answer to a problem by checking units, estimating an answer, or comparing the answer with what would be expected from common sense.

5 Using Graphs

This chapter has not yet been imported into the document. The glossary references that the chapter will include are listed below, so that links from the glossary back to the text will not cause errors.

axis

bar chart

best-fit line

constant of proportionality

dependent variable

directly proportional

exponential decay

exponential growth

6

Angles and trigonometry

This chapter has not yet been imported into the document. The glossary references that the chapter will include are listed below, so that links from the glossary back to the text will not cause errors.

acute angle

adjacent

arc

arccosine

arcsec

arcsine

arctangent

concentric

7 Logarithms

This chapter has not yet been imported into the document. The glossary references that the chapter will include are listed below, so that links from the glossary back to the text will not cause errors.

common logarithm

exponential function

logarithm

logarithm to base 10

logarithm to base e

log-linear graph

log-log graph

natural logarithm

8 Probability and descriptive statistics

This chapter has not yet been imported into the document. The glossary references that the chapter will include are listed below, so that links from the glossary back to the text will not cause errors.

accurate

addition rule for probabilities

arithmetic mean

estimated standard deviation of a population

mean

median

mode

multiplication rule for probabilities

normal distribution

population

precise

probability

random uncertainty

ratio

sample standard deviation

sample

skewed

standard deviation

systematic uncertainty

true mean

9 Statistical hypothesis testing

This chapter has not yet been imported into the document. The glossary references that the chapter will include are listed below, so that links from the glossary back to the text will not cause errors.

absolute value

alternative hypothesis

categorical level

χ^2 test

contingency-table

correlation

correlation coefficient

critical value

degrees of freedom

hypothesis

interval level

level of measurement

matched samples

null hypothesis

ordinal level

significance level

Spearman rank correlation coefficient (r_s)

statistically significant

t-test

test of association

test of difference

test statistic

unmatched samples

Differentiation 10

This chapter has not yet been imported into the document. The glossary references that the chapter will include are listed below, so that links from the glossary back to the text will not cause errors.

calculus

chord

derivative

derived function

differentiation

first derivative

second derivative

tangent

A

Resolving vectors

component

scalar

vector

modulus

Glossary

B

absolute-value The absolute value of a number is the number given without its + or – sign.

accurate Description of a set of measurements for which the systematic uncertainty is small. Compare with precise.

acute-angle An angle of less than 90°.

addition rule for probabilities A rule stating that if several possible outcomes are mutually exclusive, the probability of one or other of these outcomes occurring is found by adding their individual probabilities.

adjacent (trigonometry) The side other than the hypotenuse which is next to a particular angle in a right-angled triangle.

algebra The process of using symbols, usually letters, to represent quantities and the relationships between them.

alternative hypothesis The logical 'mirror image' of the null hypothesis proposed at the start of a statistical hypothesis test (e.g. that the means of two populations are not identical, $\mu_1 \neq \mu_2$).

arc A portion of a curve, particularly a portion of the circumference of a circle.

arccosine See inverse cosine.

arcsec An abbreviation for 'second of arc'. A 60th part of a minute of arc i.e. a 3600th part of a degree (of arc).

arcsine See inverse sine.

arctangent See inverse tangent.

arithmetic mean Measure of the average of a set of numbers. For a set of n measurements of a quantity x, the arithmetic mean $\overline{x}$ (often abbreviated to 'the mean') is defined as the sum of all the measurements divided by the total number of measurements:

$$\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

See also the true mean.

arithmetic operations The operations of addition, subtraction, multiplication and division.

axis (of a graph) A horizontal or vertical reference line which carries a set of divisions. In the case of a bar chart the divisions may be a list of categories. In the case of a graph the divisions indicate a linear or logarithmic scale, and are used to locate points on the graph.

bar chart A diagrammatic method of presenting data grouped into discrete categories. The categories are listed along one axis (usually the horizontal axis), and each category is represented by a bar (usually vertical). The bars are separated by gaps, and their height (or length) is directly proportional to the number or percentage of things or events in each category. Compare with histogram

base number When using exponents, the quantity that is raised to a power, e.g. 5 is the base in the statement $5 \times 5 \times 5 = 5^3$ and a is the base in the statement $a^3 \times a^4 = a^7$.

best-fit line A line (usually a straight line) drawn on a graph and chosen to be the best representation of the data as a whole. A best-fit line need not necessarily go through any of the data points (although it will typically go through some of them), and should be drawn in such a way that there are approximately the same number of data points above and below the line.

calculus The branch of mathematics which includes differentiation and integration.

cancellation The process of dividing both the numerator and denominator of a fraction by the same quantity. With numbers it may be quicker to use cancellation than to work out the value of the numerator and denominator separately, e.g.

$$\frac{5 \times \cancel{13}}{\cancel{13} \times 8} = \frac{5}{8}$$

Cancellation is also useful in simplifying algebraic expressions or units, e.g.

$$\frac{\cancel{a}bc^2}{2\cancel{a}d} = \frac{bc^2}{2d}$$

$$\frac{1\ \mathrm{N}\,\cancel{\mathrm{m}}}{1\ \mathrm{kg} \times 1\ \cancel{\mathrm{m}}} = \frac{1\ \cancel{\mathrm{kg}}\,\mathrm{m\,s^{-2}}}{1\ \cancel{\mathrm{kg}}} = 1\ \mathrm{m\,s^{-2}}$$

categorical level A level of measurement in which the data comprise distinct non-overlapping classes that cannot logically be ranked (e.g. presence versus absence, male versus female). See also ordinal level, interval level.

centi A prefix, used with units, to denote hundredths, and indicated by the symbol c. Thus one centimetre, denoted 1 cm, is the hundredth part of a metre. Centi is not one of the recognized submultiples in the system of SI units, but is nevertheless in common use, especially in association with units of length and volume.

χ^2 **test** (chi-squared test) A statistical hypothesis test used to determine whether there is a statistically significant association between two categorical level variables.

chord A line drawn between two points on a curve.

common denominator The same number or term occurring as the denominator of two or more fractions. For example, the numerical fractions $\frac{5}{16}$ and $\frac{7}{16}$ have the common denominator 16. It is often necessary to use equivalent fractions in order to find common denominators: for example $\frac{2}{5}$ ($= \frac{6}{15} = \frac{12}{30}$) and $\frac{8}{15}$ ($= \frac{16}{30}$) have common denominators 15 and 30 (as well as many other numbers). The algebraic fractions $\frac{a}{b}$ and $\frac{c}{d}$ have the common denominator $b \times d$.

common logarithm See logarithm to base 10.

commutative An operation for which the result is unchanged if the order of terms is reversed is described as commutative. Only two of the arithmetic operations are commutative: addition ($a + b = b + a$) and multiplication ($a \times b = b \times a$).

complex number A number of the form $n + mi$, where n is any real number, m is any non-zero real number, and $i = \sqrt{-1}$.

component (of a vector) The component of a vector along a chosen axis is obtained by drawing a line from the head of the arrow representing the vector onto the axis, such that the line meets the axis in a right angle. For example, the x-component of a vector $\boldsymbol{a}$ is $a_x = a \cos\theta$ where a is the magnitude of the vector and θ is the angle between the x-axis and the direction of the vector.

concentric Two circles are described as being concentric if they have the same centre.

constant of proportionality The constant factor that is required to turn a proportionality into an equation. The direct proportionality of $y \propto x$ can be written as $y = kx$, where k is the constant of proportionality.

contingency table A table drawn up as part of a χ^2 test in which ‘observed’ (O_i) and ‘expected’ (E_i) numbers are compared. Contingency tables may be extended by inclusion of columns for $(O_i - E_i)$, $(O_i - E_i)^2$ and $\dfrac{(O_i - E_i)^2}{E_i}$.

conversion factor The number by which one needs to divide or multiply in order to convert from one unit to another.

correlation Two variables at ordinal level or interval level are said to be correlated if, as the value of one variable increases, the value of the second variable either increases (i.e. positive correlation) or decreases (i.e. negative correlation). If the values of the two variables increase precisely in step with one another, the positive correlation can be described as ‘perfect’. In a ‘perfect’ negative correlation, the value of one variable decreases precisely as the other increases. Correlations may or may not be statistically significant.

correlation coefficient The correlation coefficient (r) of a ‘perfect’ positive correlation is $+1$, while that of a ‘perfect’ negative correlation is -1. When there is complete lack of correlation between two variables, $r = 0$. For a positive correlation that is less than ‘perfect’, $1 > r > 0$. For a negative correlation that is less than ‘perfect’, $0 > r > -1$.

cosine The cosine of an angle θ in a right-angled triangle is defined by

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

where ‘adjacent’ is the length of the side adjacent to θ and ‘hypotenuse’ is the length of the hypotenuse.

critical value At a particular number of degrees of freedom (in many statistical hypothesis tests), the critical value is the most extreme (usually the largest, but in some statistical tests the smallest) value that the test statistic is expected to have for a particular significance level.

deci Prefix, used with units, to denote tenths, and indicated by the symbol d. Thus one decibel, denoted 1 dB, is equal to one tenth of a bel. Deci is not one of the recognized submultiples in the system of SI units, but is commonly used in certain areas: for example the concentration of a chemical dissolved in a solvent is often expressed in units of moles per decimetre cubed (mol dm^{-3}).

decimal notation Method of representing numbers, according to which the integral and fractional parts of a number are separated by a decimal point. The decimal point is written as a full stop, with the integral part of the number to the left of it. The first digit after the decimal point indicates the number of tenths, the second indicates the number of hundredths, the third the number of thousandths, etc.

decimal places See places of decimals.

degree (of arc) A 360th of a complete revolution.

degree-Celsius An everyday unit of temperature, given the symbol °C. Pure water freezes at 0 °C and boils at 100 °C. Temperatures may be converted from degrees Celsius to the SI unit of temperature, kelvin, using the word equation (temperature in kelvin) = (temperature in degrees Celsius) + 273.15

degrees of freedom A device used in many statistical hypothesis tests to allow for the fact that the more data that are collected, the more scope there is for the test statistic to deviate from the value expected (generally, zero) if the null hypothesis were true.

denominator The number or term on the bottom of a fraction. For example, in the fraction $\frac{1}{2\pi}$, the denominator is 2π; in the fraction $\frac{mn}{pq}$, the denominator is pq. See also: numerator.

dependent variable A quantity whose value is determined by the value of one or more other variables. On a graph, the dependent variable is, by convention, plotted along the vertical axis. Compare with: independent variable.

derivative The derivative (or derived function) of a function $f(x)$ with respect to x is another function of x that is equal to the rate of change of $f(x)$ with respect to x. Its value at any given value of x is equal to the ratio $\frac{\Delta f}{\Delta x}$ in the limit as Δx becomes very small, and is usually written as $\frac{\mathrm{d}f}{\mathrm{d}x}$ or $f'(x)$. The value of $\frac{\mathrm{d}f}{\mathrm{d}x}$ at each value of x is also equal to the gradient of the graph of f plotted against x at that value of x. A derivative of the type is sometimes called the first derivative to distinguish it from the second derivative of the function.

derived function See derivative.

differentiation A mathematical process that enables the derivative of a function to be determined.

directly proportional (quantities) Two quantities x and y are said to be directly proportional to each other if multiplying (or dividing) x by a certain amount automatically results in y being multiplied (or divided) by the same amount. Direct proportionality between x and y is indicated by writing $y \propto x$. The direct proportionality can also be written as an equation of form $y = kx$, where k is a constant called the constant of proportionality. A graph in which y is plotted against x will be a straight line with gradient equal to k. See also inversely proportional.

elimination A method of combining two or more equations by eliminating variables that are common to them.

equation An expression containing an equals sign. What is written on one side of the equation must always be equal to what is written on the other side.

equivalent fractions Fractions that have the same value, e.g. $\frac{2}{3}$, $\frac{4}{6}$, $\frac{8}{12}$, $\frac{20}{30}$, etc.

estimated standard deviation of a population The best estimate that can be made for the standard deviation of some quantity for a whole population. This estimate is usually set equal to s_{n-1}, which is calculated from measurements of the quantity made on an unbiased sample drawn from the population. If the sample consists of n members and the quantity x is measured once for each member, then

$$s_{n-1} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2}$$

where $\overline{x}$ is the arithmetic mean of the measurements. The symbol σ_{n-1} is also widely used (especially on calculators) as an equivalent to s_{n-1}.

evaluate An instruction to work out the value of an expression.

exponent When raising quantities to powers, the number to which a quantity is raised, e.g. in the term 2^3, the exponent is 3.

exponential decay Decay in which the time taken for a quantity to fall to half its original value is always the same; this time is known as the half-life. A quantity N with an initial value of N_0 at time $t = 0$ decays exponentially if $N = N_0 e^{-\lambda t}$, where λ is a constant known as the decay constant.

exponential function A function of the type $y = Ce^{kx}$ where C and k are constants. A function of this type has the property that $\frac{dy}{dx}$ is proportional to y.

exponential growth Growth in which the quantity being measured increases by a constant factor in any given time interval. A quantity n with a starting value of n_0 at time $t = 0$ grows exponentially if $n = n_0 e^{at}$, where a is a positive constant.

expression A combination of variables (such as $a_x t$ or $u_x + a_x t$). Unlike an equation, an expression is unlikely to contain an equals sign.

extrapolation The process of extending a graph beyond the highest or lowest data points in order to find the values of one or both of the plotted quantities outside the original range within which data were obtained.

factor A term which when multiplied to other terms results in the original expression, so 6 and 4 are factors of 24 and $(a - 3)$ and $(a + 5)$ are factors of $a^2 + 2a - 15$.

factorize To find the factors of an expression.

first derivative See derivative.

formula A rule expressed in algebraic symbols.

fraction A number expressed in the form of one integer divided by another, e.g.

$\frac{1}{4}$; $\frac{3}{8}$; $\frac{21}{13}$. One algebraic term divided by another may also be described as a fraction. See also: improper fraction, mixed number, equivalent fractions, numerator and denominator.

function If the value of a variable f depends on the value of another variable x, then f is said to be a function of x and is written as $f(x)$. In general, there is only one value of $f(x)$ for each value of x.

gradient (of a graph) The slope of a line on a graph. The gradient is a measure of how rapidly the quantity plotted on the vertical axis changes in response to a change in the quantity plotted on the horizontal axis. If the graph is a straight line, then the gradient is the same at all points on the line and may be calculated by dividing the vertical 'rise' between any two points on the line by the horizontal 'run' between the same two points. If the graph is a curved line, the gradient at any point on the curve is defined by the gradient of the tangent to the curve at that point. See also: derivative.

graph A method of illustrating the relationship between two variable quantities by plotting the measured values of one of the quantities using a linear or logarithmic scale along a horizontal axis, and the measured values of the other quantity using a linear or logarithmic scale along a vertical axis. See also: dependent variable, independent variable, sketch graph.

half-life The time taken for half the nuclei in a radioactive sample to decay. See also exponential decay.

histogram A diagrammatic method of presenting data, in which the horizontal axis is divided into (usually equal) intervals of a continuously variable quantity. Rectangles of width equal to the interval have a height scaled to show the value of the quantity plotted on the vertical axis that applies at the particular interval. For example, the intervals could be the months in the year and the vertical axis could represent the mean (monthly) rainfall in millimetres. Compare with bar chart.

hyperbola A curve, part of which may be obtained by plotting inversely proportional quantities against each other on a .

hypotenuse The side opposite to the right-angle in a right-angled triangle.

hypothesis A plausible idea tentatively put forward to explain an observation. Traditionally, a hypothesis is tested by making predictions that would follow if the hypothesis is correct. If these predictions are borne out by experiment or further observation, then this lends weight to the hypothesis *but does not prove it to be correct*. If the predictions are not borne out, then the hypothesis is either rejected or modified.

imaginary number A number of the form mi, where m is any non-zero real number and $i = \sqrt{-1}$.

improper fraction A fraction in which the numerator is greater than the denominator, e.g. $\frac{12}{7}$. An improper fraction may also be written as a mixed number.

independent variable The quantity in an experiment or mathematical manipulation whose value(s) can be chosen at will within a given range. On a graph, the independent variable, is by convention, plotted along the horizontal axis. Compare with dependent variable.

index (plural indices) See exponent.

integer A positive or negative whole number (including zero).

integral Pertaining to an integer. For example the statement that m can take integral values from -2 to $+2$ means that the possible values of m are -2, -1, 0, 1 and 2.

intercept The value on one axis of a graph at which a plotted straight line crosses that axis, provided that axis does pass through the zero point on the other axis. If the plotted line has an equation of form $y = mx + c$, the intercept on the y axis is equal to c.

interpolation The process of reading between data points plotted on a graph, in order to find the value of one or both of the plotted quantities at intermediate positions.

interval level A level of measurement in which the *actual* values of measurements or counts are known and used in statistical analysis (e.g. dry mass in grams, number of flowers per plant). See also categorical level, ordinal level.

inverse cosine x is the inverse cosine (arccosine) of y if x is the angle whose cosine is y. i.e. $x = \cos^{-1} y$ ($x = \arccos y$) if $y = \cos x$.

inverse sine x is the inverse sine (arcsine) of y if x is the angle whose sine is y. i.e. $x = \sin^{-1} y$ ($x = \arcsin y$) if $y = sinx$.

inverse tangent x is the inverse tangent (arctangent) of y if x is the angle whose tangent is y, i.e. $x = \tan^{-1} y$ ($x = \arctan y$) if $y = \tan x$.

inverse trigonometric function If y is a trigonometric ratio of the angle x, then x is the inverse trigonometric function of y. For example, if $y = \sin x$, the inverse trigonometric function is $x = \sin^{-1} y$ (or $\arcsin y$) where $\sin^{-1} y$ ($\arcsin y$) is the angle whose sine is y.

inversely proportional (quantities) Two quantities x and y are said to be inversely proportional to each other if an increase in x by a certain factor automatically results in a decrease in y by the same factor (e.g. if the value of x doubles, then the value of y halves). Inverse proportionality between x and y is indicated by writing $y \propto \frac{1}{x}$. A graph in which y is plotted against x will be a hyperbola. See also: directly proportional.

irrational number A number that cannot be obtained by dividing one integer by another, e.g. π, $\sqrt{2}$ and e. See also rational number.

latitude Part of the specification of the position of a point on the Earth's surface: the distance north or south of the Equator measured in degrees. A line of latitude is an imaginary circle on the surface of the Earth.

level of measurement The three levels of measurement that data may be known or analysed at are categorical level, interval level or ordinal level.

linear scale A scale on which the steps between adjacent divisions correspond to the addition or subtraction of a fixed quantity.

logarithm The logarithm of a number to a given base is the power to which the base must be raised in order to produce the number.

logarithm to base 10 The logarithm to base 10 (or 'common logarithm', $\log_{10}$) of p is the power to which 10 must be raised in order to equal p. i.e. if $p = 10^n$, then $\log_{10} p = n$.

logarithm to base e The logarithm to base e (or 'natural logarithm') of p is the power to which e must be raised in order to equal p, i.e. if $p = \mathrm{e}^q$, then $\ln p = q$.

logarithmic scale Scale on which the steps between adjacent divisions correspond to multiplication or division by a fixed amount, usually a power of ten.

log-linear graph A graph of the logarithm of one quantity against the actual value of another quantity. For an exponential function of the type $y = C\mathrm{e}^{kx}$, graphs of $\log_{10} y$ against x and of $\ln y$ against x will both be straight lines.

log-log graph A graph of the logarithm of one quantity against the logarithm of another quantity. For a function of the type $y = ax^b$ (e.g. $y = 2x^3$) graphs of $\log_{10} y$ against $\log_{10} x$ and of $\ln y$ against $\ln x$ will both be straight lines.

longitude Part of the specification of the position of a point on the Earth's surface. A line of longitude is an imaginary semicircle that runs from one pole to the other. The line of zero longitude passes through Greenwich in London. Other lines of longitude are specified by the angle east or west of the line of zero longitude.

lowest common denominator The smallest common denominator of two or more fractions.

magnitude The size of a quantity, also referred to as the 'modulus'. Vector quantities have both magnitude and direction; scalar quantities have only magnitude.

matched samples When data are collected from two samples such that each item of data from one sample can be uniquely matched with just one item of data from the other sample (e.g. blood glucose levels measured in individuals before and after they have taken medication), the samples are described as matched. See also unmatched samples.

mean Term commonly used as an abbreviation for arithmetic mean.

median The middle value in a series when the values are arranged in either increasing or decreasing order. If the series contains an odd number of items, the median is the value of the middle item; if it contains an even number of items, the median is the arithmetic mean of the values of the middle two items.

minute (of arc) A 60th part of an degree (of arc).

mixed number A number consisting of a non-zero integer and a fraction, e.g. $3\frac{1}{2}$.
Any improper fraction may also be written as a mixed number: for example $\frac{8}{3} = 2\frac{2}{3}$.

mode The most frequently occurring value in a set of data.

modulus See magnitude.

multiplication rule for probabilities A rule stating that if a number of outcomes occur independently of one another, the probability of them all happening together is found by multiplying the individual probabilities.

natural logarithm See logarithm to base e.

normal distribution Distribution of measurements or characteristics which lie on a bell-shaped curve that is symmetric about its peak, with the peak corresponding to the mean value. Repeated independent measurements of the same quantity approximate to a normal distribution, as do quantitative characters in natural populations (e.g. height in human beings).

null hypothesis A 'no difference' hypothesis proposed at the start of a statistical hypothesis test (e.g. that the means of two populations are identical, $\mu_1 = \mu_2$). Compare with alternative hypothesis.

numerator The number or term on the top of a fraction. For example, in the fraction $\frac{3}{4}$, the numerator is 3; in the fraction $\frac{a+b}{c}$, the numerator is $a + b$. See also denominator.

opposite (trigonometry) The side opposite to a particular angle in a right-angled triangle.

order of magnitude The approximate value of a quantity, expressed as the nearest power of ten. If the value of the quantity is expressed in scientific notation as $a \times 10^n$, then the order of magnitude of the quantity is 10^n if $a < 5$ and 10^{n+1} if $a > 5$. The phrase is also used to compare the sizes of quantities, as in 'a metre is three orders of magnitude longer than a millimetre' or 'a picogram is twelve orders of magnitude smaller than a gram'.

ordinal level A level of measurement in which the data can be logically ranked but in which the *actual* values of the measurements or counts are either not known or not used in statistical analysis (e.g. tallest to shortest, heaviest to lightest). See also categorical level, interval level.

origin (of a graph) The point on a graph at which the quantities plotted on the horizontal axis and the vertical axis are both zero.

parabola A curve that may be described by an equation of the form $y = ax^2 + bx + c$, where x and y are variables, a is a non-zero constant, and b and c are constants that may take any value.

percentage A way of expressing a fraction with a denominator of 100. For example, 12 per cent (also written 12%) is equivalent to twelve parts per hundred or $\frac{12}{100}$.

places of decimals In decimal notation, the number of digits after the decimal point (including zeroes). Thus 21.327 and 3.000 are both given to three places of decimals.

population Statistical term used to describe the complete set of things or events being studied.

power See exponent.

powers of ten notation A method of representing a number as a larger or smaller number multiplied by ten raised to the appropriate power. For example, 2576 can be written in powers of ten notation as 25.76×10^2 or 2.576×10^3, or 0.02576×10^5 or 257600×10^{-2}. See also scientific notation.

precise Description of a set of measurements for which the random uncertainty is small. Compare with accurate.

probability If a process is repeated a very large number if times, then the probability of a particular outcome may be defined in terms of results obtained as the fraction of results corresponding to that particular outcome. If the process has n equally likely outcomes and q of those outcomes correspond to a particular event, then the probability of that event is defined as q/n. There are, for example, 6 equally likely outcomes for the process of rolling a fair die. Only one of those outcomes corresponds to the event 'throwing a six', so the probability of throwing a six is $\frac{1}{6}$. Five of the outcomes correspond to the event 'not throwing a six', so the probability of not throwing a six is $\frac{5}{6}$.

product The result of a multiplication operation. For example, the product of 3 and 5 is 15.

proportional See directly proportional, inversely proportional.

Pythagoras' Theorem The square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides.

quadratic equation An algebraic equation for x of the form $ax^2 + bx + c = 0$, where $a \neq 0$ and b and c can take any value. For example, $2x^2 - x + 3 = 0$ is a quadratic equation.

quadratic equation formula The solutions of a quadratic equation of the form $ax^2 + bx + c = 0$ are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

radian The angle subtended at the centre of a circle by an arc equal in length to the radius. In general, the angle θ subtended by an arc length s in a circle of radius r is given by θ (in radians) $= \dfrac{s}{r}$.

random uncertainty Measured values of one quantity that are scattered over a limited range about a mean value are said to be subject to random uncertainty. The larger the random uncertainty associated with the measurements, the larger will be the scatter. See also precise and systematic uncertainty.

ratio The relationship between the sizes of two comparable quantities. For example, if a group of 11 people is made up of 8 women and 3 men, the ratio of women to men is said as 8 to 3 and written as 8 : 3. Ratios may be fairly easily converted into fractions. In this particular example $\frac{8}{8+3} = \frac{8}{11}$ of the group are women and $\frac{3}{11}$ are men.

rational number Any number that can be written in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$, e.g. $7 = \frac{7}{1}$; $-6 = \frac{-6}{1}$; $-\frac{1}{3}$; $3.125 = \frac{25}{8}$. Every terminating or recurring decimal is a rational number. See also: irrational number.

real number A number that can be placed on the number line. The set of real numbers is made up of all the rational and irrational numbers.

reciprocal A term that is related to another as $\frac{2}{3}$ is related to $\frac{3}{2}$. The reciprocal of $\frac{y}{x}$ is $\frac{x}{y}$, and vice versa, for any non-zero values of x and y. The reciprocal of N^m is N^{-m} and vice versa.

recurring decimal A number in which the pattern of digits after the decimal point repeats itself indefinitely. Every recurring decimal is a rational number and can therefore be written as a fraction, e.g. $0.3333\ldots = \frac{1}{3}$; $0.123\,123\,123\ldots = \frac{41}{333}$; $0.2345\,2345\,2345\ldots = \frac{2345}{9999}$.

right angle The angle between two directions that are perpendicular (i.e. at $90°$) to each other.

right-angled triangle A triangle where the angle between two of the sides is a right angle.

rounding error An error introduced into a calculation by working to too few significant figures. To avoid rounding errors you should work to at least one more significant figure than is required in the final answer, and just round at the end of the whole calculation.

sample Statistical term used to describe an unbiased sub-set of a population.

sample standard deviation See estimated standard deviation of a population.

scalar A quantity with magnitude but no direction. Compare with vector.

scientific notation Method of writing numbers, according to which any rational number can be written in the form $a \times 10^n$ where a is either an integer or a number written in decimal notation, $1 \le a < 10$, and n is an integer. Thus $5\,870\,000$ may be written in scientific notation as 5.87×10^6, and $0.003\,261$

may be written in scientific notation as 3.261×10^{-3}. The terms 'standard form' and 'standard index form' are equivalent to the term scientific notation.

second (of arc) See arcsec.

second derivative A derivative of a derivative, for example the derivative of $\frac{df}{dx}$ with respect to x. A second derivative is usually written as or $\frac{d^2 f}{dx^2}$ or $f''(x)$.

SI units An internationally agreed system of units. In this system, there are seven base units (which include the metre, kilogram and the second) and an unlimited number of derived units obtained by combining the base units in various ways. The system recognizes a number of standard abbreviations (of which SI, standing for Système International, is one). The system also uses certain standard multiples and submultiples, represented by standard prefixes. See also centi and deci.

significance level The probability that the value of a test statistic could be as extreme (usually as large, but in some statistical tests as small) as the value obtained in a statistical hypothesis test if the null hypothesis were true.

significant figures The number of digits, excluding leading zeroes, quoted for the value of a quantity, and defined as the number of digits known with certainty plus one uncertain digit. Thus if a measured temperature is given as 23.7°C (i.e. quoted to three significant figures) this implies that the first two digits are certain, but there is some uncertainty in the final digit, so the real temperature might be 23.6°C or 23.8°C. The larger the number of significant figures quoted for a value, the smaller is the uncertainty in that value. Leading zeroes in decimal numbers do not count as significant figures (e.g. 0.002 45 is expressed to three significant figures). Numbers equal to or greater than 100 can be unambiguously expressed to two significant figures only by the use of scientific notation (e.g. 450 can only be unambiguously expressed to two significant figures by writing it in the form 4.5×10^2). Similarly, scientific notation must be used to express numbers equal to or greater than 1000 unambiguously to 3 significant figures.

similar Two triangles (or other objects) are described as being similar if they have the same shape but different size.

simplify To write an equation or expression in its simplest form.

simultaneous equations Two or more equations which must hold true simultaneously.

sine The sine of an angle *q* in a right-angled triangle is defined by

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

where ‘opposite’ is the length of the side opposite θ and ‘hypotenuse’ is the length of the hypotenuse.

sketch graph A graph drawn to illustrate the nature of the relationship between quantities, but not involving accurate plotting. On a sketch graph the origin is usually indicated, but the axes are not scaled.

skewed Description of distributions that are not symmetric about their mean value.

small angle approximation For small angles (less than about 0.1 radian) $\cos\theta \approx 0$, and if the angle is stated in radians, $\sin\theta \approx \theta$, $\tan\theta \approx \theta$.

solution The answer, especially numerical value or values which satisfy an algebraic equation.

solve To find an answer, usually to find the numerical values which satisfy an algebraic equation.

Spearman rank correlation coefficient (r_s) A test statistic calculated in a statistical hypothesis test used to determine whether or not there is a statistically significant correlation between two ordinal level variables.

square root The number or expression that multiplied by itself gives N is called the square root of N. The positive square root of N can be written as either $\sqrt{N}$ or $N^{\frac{1}{2}}$.

standard deviation A quantitative measure of the spread of a set of measurements. For n repeated measurements of a quantity, with arithmetic mean $\overline{x}$, the standard deviation s_n is given by

$$s_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2}$$

The symbol σ_n is also widely used (especially on calculators) as an equivalent to s_n. See also: sample standard deviation, estimated standard deviation of a population.

standard form See scientific notation.

standard index form See scientific notation.

statistically significant In science, the result of a statistical hypothesis test is conventionally regarded as statistically significant if the probability of the value of the test statistic being as large (or, in some statistical tests, as small) as the one obtained is less than 0.05.

subject The term written by itself, usually to the left of the equals sign in a mathematical equation.

subtend A straight line rotating about a certain point is said to subtend the angle it passes through.

sum The result of an addition operation. For example, the sum of 3 and 2 is 5. A summation sign may be used as shorthand for more complicated addition operations, e.g.

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \ldots + x_n.$$

systematic uncertainty Measured values of one quantity that are consistently too large or too small because of bias in the measuring instrument or the measurement technique are said to be subject to systematic uncertainty. See also accurate, random uncertainty.

t-test One of a number of statistical tests of a hypothesis used to determine whether there is a statistically significant difference between the estimated population means calculated from two samples. Different versions of the test are available for matched samples and unmatched samples.

tangent (to a curved graph) The tangent to a curve at a given point P is the straight line that just touches the curve at P and has the same gradient as the curve at the point P.

tangent (trigonometry) The tangent of an angle θ in a right-angled triangle is defined by

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

where ‘opposite’ is the length of the side opposite and ‘adjacent’ is the length of the side adjacent to θ.

term A single variable (such as v_x or u_x in the equation $v_x = u_x + a_x t$) or a combination of variables, such as $a_x t$.

test of association A statistical hypothesis test used to determine whether there is a statistically significant association between two categorical level variables (e.g. χ^2 test) or a statistically significant correlation between two variables at ordinal level (e.g. Spearman rank correlation (r_s)) or at interval level (other correlation coefficients (r)).

test of difference A statistical hypothesis test used to test whether there is a statistically significant difference between, for example, the estimated population means (e.g. t-tests) or estimated population medians (other tests) calculated from two samples.

test statistic In most statistical tests of a hypothesis, the value of a test statistic is calculated using an equation. The value of the test statistic is then compared with a table of critical values in order to determine whether the null hypothesis ought to be accepted or rejected at a particular significance level.

trigonometric ratios The ratios of the sides of a right-angled triangle, including tangent, sine, cosine.

trigonometry The branch of mathematics which deals with the relations between the sides and angles of triangles, usually right-angled triangles.

true mean The arithmetic mean of some quantity for a whole population, usually denoted by the symbol μ. For a large population, the true mean is generally unknowable and the best estimate that can be made of it is the mean of the quantity for an unbiased sample drawn from the population.

unmatched samples When data are collected from two samples such that there is no logical connection between any particular item of data from one sample and any particular item of data from the other sample (e.g. the heights of plants randomly assigned to either an experimental or a control group), the samples are described as unmatched. See also matched samples.

variable A quantity that can take a number of values.

vector A physical quantity that has a definite magnitude and points in a definite direction.

word equation An equation in which the quantities under consideration are described in words.

Hidden material

This ‘chapter’ contains material which you won’t normally read through in sequence, but will access it through the links from the main text.

Question 1.1 (a)

$(-3) \times 4 = -12$

Question 1.1 (b)

$$(-10) - (-5) = -5$$

Question 1.1 (c)

$6 \div (-2) = -3$

Question 1.1 (d)

$(-12) \div (-6) = 2$

Question 1.2

The lowest temperature in the oceans, which corresponds to the freezing point, is 31.9 Celsius degrees colder than the highest recorded temperature, which is 30.0 °C.

$$\begin{aligned}\text{Therefore, freezing point of seawater} &= 30.0\ ^\circ\text{C} - 31.9\ ^\circ\text{C} \\ &= -1.9\ ^\circ\text{C}\end{aligned}$$

Question 1.3 (a)

$$117 - (-38) + (-286) = -131$$

Question 1.3 (b)

$(-1624) \div (-29) = 56$

Question 1.3 (c)

$$(-123) \times (-24) = 2952$$

Question 1.4 (a)

The lowest common denominator is 6, so

$$\frac{2}{3}-\frac{1}{6}=\frac{2\times 2}{3\times 2}-\frac{1}{6}=\frac{4}{6}-\frac{1}{6}=\frac{3}{6}$$

Dividing top and bottom by 3 gives

$$\frac{3}{6}=\frac{1}{2}$$

Alternatively,

$$\frac{2}{3}-\frac{1}{6}=\frac{2\times 6}{3\times 6}-\frac{1\times 3}{6\times 3}=\frac{12}{18}-\frac{3}{18}=\frac{9}{18}$$

Dividing top and bottom by 9 gives

$$\frac{9}{18}=\frac{1}{2}$$

as before.

Question 1.4 (b)

The lowest common denominator is 30, so

$$\begin{aligned}\frac{1}{3}+\frac{1}{2}-\frac{2}{5} &= \frac{1\times 10}{30}+\frac{1\times 15}{30}-\frac{2\times 6}{30}\\ &= \frac{10}{30}+\frac{15}{30}-\frac{12}{30}\\ &= \frac{13}{30}\end{aligned}$$

Question 1.4 (c)

In this case, the lowest common denominator isn't immediately obvious, but a common denominator will certainly be given by the product of 3 and 28, so

$$\begin{aligned}\frac{5}{28}-\frac{1}{3} &= \frac{5\times 3}{28\times 3}-\frac{1\times 28}{3\times 28}\\ &= \frac{15}{84}-\frac{28}{84}\\ &= -\frac{13}{84}\end{aligned}$$

Question 1.5 (a)

The original fraction, $\frac{4}{16} = \frac{1}{4} = 0.25$.

You may have chosen any number for your calculations. In this answer the number 2 is used, but the principles hold good whatever choice of (non-zero) number is made.

Suppose we were to add 2 to the numerator and to the denominator

$$\frac{4+2}{16+2} = \frac{6}{18} = 0.333 \text{ to three places of decimals}$$

This is not the same as the original fraction. (There is just one special case in which this kind of operation would not change the value of the fraction and that is adding 0 to top and bottom, which obviously leaves the fraction unchanged.)

Question 1.5 (b)

Suppose we were to subtract 2 from the numerator and from the denominator

$$\frac{4-2}{16-2} = \frac{2}{14} = 0.143 \text{ to three places of decimals}$$

This is not the same as the original fraction. (Again, subtracting 0 from top and bottom is the only case in which this operation leaves the fraction unchanged.)

Question 1.5 (c)

If we square the numerator and the denominator

$$\frac{4 \times 4}{16 \times 16} = \frac{16}{256} = 0.0625$$

This is not the same as the original fraction.

Question 1.5 (d)

If we take the square root of the numerator and of the denominator

$$\frac{\sqrt{4}}{\sqrt{16}} = \frac{2}{4} = 0.5$$

This is not the same as the original fraction.

Incidentally, checking a general rule by trying out a specific numerical example is a helpful technique, which will be useful for algebra in Chapter 4.

Question 1.6 (a)

$$\frac{2}{7} \times 3 = \frac{2 \times 3}{7} = \frac{6}{7}$$

Question 1.6 (b)

$$\frac{5}{9} \div 7 = \frac{5}{9} \times \frac{1}{7} = \frac{5 \times 1}{9 \times 7} = \frac{5}{63}$$

Question 1.6 (c)

$$\frac{1/6}{1/3} = \frac{1}{6} \div \frac{1}{3} = \frac{1}{6} \times \frac{3}{1} = \frac{3}{6} = \frac{1}{2}$$

Question 1.6 (d)

$$\frac{3}{4} \times \frac{7}{8} \times \frac{2}{7} = \frac{3 \times 7 \times 2}{4 \times 8 \times 7} = \frac{42}{224}$$

Dividing top and bottom by 2, and then by 7

$$\frac{42}{224} = \frac{21}{112} = \frac{3}{16}$$

Alternatively, the original could have been simplified in the same way before carrying out any multiplication:

$$\frac{3}{\not{4}_{2}} \times \frac{\not{7}^{1}}{8} \times \frac{\not{2}^{1}}{\not{7}_{1}} = \frac{3}{16}$$

Question 1.7 (a)

$$2^{-2} = \frac{1}{2^2} = \frac{1}{2 \times 2} = \frac{1}{4}$$

You might have gone one step further and expressed this in decimal notation as 0.25.

Question 1.7 (b)

$$\frac{1}{3^{-3}} = 3^3 = 3 \times 3 \times 3 = 27$$

Question 1.7 (c)

$$\frac{1}{4^0} = \frac{1}{1} = 1$$

Question 1.7 (d)

$$\frac{1}{10^4} = \frac{1}{10\,000} = 0.000\,1$$

Question 1.8 (a)

$2^9 = 512$

Question 1.8 (b)

$$3^{-3} = \frac{1}{3^3} = 0.037 \text{ to three places of decimals}$$

It doesn't matter if you quoted more digits in your answer than this. There is more explanation in Chapter 2 about how and when to round off the values given on your calculator display.

Question 1.8 (c)

$$\frac{1}{4^2} = 4^{-2} = 0.0625$$

Question 1.9 (a)

$2^{30} \times 2^{2} = 2^{(30+2)} = 2^{32}$

Question 1.9 (b)

$$3^{25} \times 3^{-9} = 3^{(25+(-9))} = 3^{16}$$

Question 1.9 (c)

$10^2/10^3 = 10^2 \div 10^3 = 10^{(2-3)} = 10^{-1}$ (or 1/10)

Question 1.9 (d)

$$10^2/10^{-3} = 10^2 \div 10^{-3} = 10^{(2-(-3))} = 10^5$$

or alternatively

$$10^2/10^{-3} = 10^2 \times \frac{1}{10^{-3}} = 10^2 \times 10^3 = 10^5$$

Question 1.9 (e)

$$10^{-4} \div 10^{2} = 10^{(-4-2)} = 10^{-6}$$

Question 1.9 (f)

$$\frac{10^5 \times 10^{-2}}{10^3} = 10^{(5+(-2)-3)} = 10^0 \text{ (or 1)}$$

Question 1.10 (a)

$$\left(4^{16}\right)^2 = 4^{16\times 2} = 4^{32}$$

Question 1.10 (b)

$$\left(5^{-3}\right)^{2} = 5^{(-3)\times 2} = 5^{-6}$$

This could also be written as $\frac{1}{5^{6}}$.

Question 1.10 (c)

$$\left(10^{25}\right)^{-1} = 10^{25\times(-1)} = 10^{-25}$$

This could also be written as $\dfrac{1}{10^{25}}$.

Question 1.10 (d)

$$\left(\frac{1}{3^3}\right)^6 = \frac{1^6}{(3^3)^6} = \frac{1}{3^{3\times6}} = \frac{1}{3^{18}}$$

or alternatively

$$\left(\frac{1}{3^3}\right)^6 = \left(3^{-3}\right)^6 = 3^{-3\times6} = 3^{-18} = \frac{1}{3^{18}}$$

Question 1.11 (a)

From Equation 1.3

$$\left(2^4\right)^{\frac{1}{2}} = 2^{\left(4\times\frac{1}{2}\right)} = 2^2 = 4$$

Question 1.11 (b)

From Equation 1.3

$$\sqrt{10^4} = \left(10^4\right)^{\frac{1}{2}} = 10^{4\times\frac{1}{2}} = 10^2 = 100$$

Question 1.11 (c)

From Equation 1.3

$$100^{\frac{3}{2}} = \left(100^{\frac{1}{2}}\right)^3 = 10^3 = 1000$$

Alternatively

$$100^{\frac{3}{2}} = \left(100^3\right)^{\frac{1}{2}} = \left(10^6\right)^{\frac{1}{2}} = 10^{6/2} = 10^3 = 1000$$

Question 1.11 (d)

$$125^{-1/3} = \frac{1}{125^{1/3}} = \frac{1}{5} = 0.2$$

Since the cube root of 125 is 5.

Question 1.12 (a)

Multiplication takes precedence over subtraction, so

$$\begin{aligned} 35 - 5 \times 2 &= 35 - (5 \times 2) \\ &= 35 - 10 \\ &= 25 \end{aligned}$$

Question 1.12 (b)

Here the brackets take precedence, so

$$\begin{aligned}(35-5)\times 2 &= 30\times 2\\ &= 60\end{aligned}$$

Question 1.12 (c)

Again, the brackets take precedence over the (implied) multiplication, so

$$\begin{aligned} 5(2-3) &= 5 \times (-1) \\ &= -5 \end{aligned}$$

Question 1.12 (d)

Here the exponent takes precedence:

$$
\begin{aligned}
3 \times 2^2 &= 3 \times 4 \\
&= 12
\end{aligned}
$$

Question 1.12 (e)

The exponent takes precedence again:

$$\begin{aligned} 2^3 + 3 &= 8 + 3 \\ &= 11 \end{aligned}$$

Question 1.12 (f)

Here both brackets take precedence over the (implied) multiplication:

$$\begin{aligned}(2+6)(1+2) &= 8\times 3\\ &= 24\end{aligned}$$

Question 2.1 (a)

$$
\begin{aligned}
5.4 \times 10^4 &= 5.4 \times 10\,000 \\
&= 54\,000
\end{aligned}
$$

Question 2.1 (b)

$$
\begin{aligned}
2.1 \times 10^{-2} &= 2.1 \times \frac{1}{100} \\
&= \frac{2.1}{100} \\
&= 0.021
\end{aligned}
$$

Question 2.1 (c)

$$
\begin{aligned}
0.6 \times 10^{-1} &= 0.6 \times \frac{1}{10} \\
&= \frac{0.6}{10} \\
&= 0.06
\end{aligned}
$$

Question 2.2 (a)

$$\begin{aligned} 215 &= 2.15 \times 100 \\ &= 2.15 \times 10^2 \end{aligned}$$

Question 2.2 (b)

$$\begin{aligned} 46.7 &= 4.67 \times 10 \\ &= 4.67 \times 10^1 \end{aligned}$$

Question 2.2 (c)

$$\begin{aligned} 152 \times 10^3 &= 1.52 \times 100 \times 10^3 \\ &= 1.52 \times 10^2 \times 10^3 \\ &= 1.52 \times 10^{(2+3)} \\ &= 1.52 \times 10^5 \end{aligned}$$

Question 2.2 (d)

$$
\begin{aligned}
0.000\,0876 &= \frac{8.76}{100\,000} \\
&= \frac{8.76}{10^5} \\
&= 8.76 \times 10^{-5}
\end{aligned}
$$

Question 2.3 (a)

A kilometre is 10^3 times bigger than a metre, so

$$\begin{aligned} 3476\text{ km} &= 3.476 \times 10^3\text{ km} \\ &= 3.476 \times 10^3 \times 10^3\text{ m} \\ &= 3.476 \times 10^6\text{ m} \end{aligned}$$

Question 2.3 (b)

A micrometre is 10^3 times bigger than a nanometre, so

$$8.0\ \mu\text{m} = 8.0 \times 10^3\ \text{nm}$$

Question 2.3 (c)

A second is 10^3 times bigger than a millisecond, so

$$0.8\ \text{s} = 0.8 \times 10^3\ \text{ms}$$

To express this in scientific notation, we need to multiply and divide the right-hand side by 10:

$$\begin{aligned} 0.8 \times 10^3\ \text{ms} &= (0.8 \times 10) \times \frac{10^3}{10}\ \text{ms} \\ &= 8 \times \left(10^3 \times 10^{-1}\right)\ \text{ms} \\ &= 8 \times 10^{(3-1)}\ \text{ms} \\ &= 8 \times 10^2\ \text{ms} \end{aligned}$$

Question 2.4 (a)

One million = 10^6, so the distance is

$$5900 \times 10^6 \text{ km} = 5.9 \times 10^9 \text{ km}$$
$$\sim 10^{10} \text{ km (or } 10^{13} \text{ m)}$$

Question 2.4 (b)

The diameter of a spherical object is given by twice its radius. So for the Sun,

$$\begin{aligned}\text{diameter} &= 2 \times 6.97 \times 10^7 \text{ m} \\ &= 13.94 \times 10^7 \text{ m} \\ &= 1.394 \times 10^8 \text{ m} \\ &\sim 10^8 \text{ m}\end{aligned}$$

Question 2.4 (c)

$$2\pi = 2 \times 3.14 \text{ (to two places of decimals)}$$
$$= 6.28$$

This is greater than 5, so can be rounded up to the next power of ten to give the order of magnitude, i.e. $2\pi \sim 10$ (or 10^1).

Question 2.4 (d)

$$
\begin{aligned}
7.31 \times 10^{-26}\ \text{kg} &\sim 10 \times 10^{-26}\ \text{kg} \\
&\sim 10^{(-26+1)}\ \text{kg} \\
&\sim 10^{-25}\ \text{kg}
\end{aligned}
$$

Question 2.5 (a)

(i) 10^0 m = 1 m and 10^{-2} m = 0.01 m, so the difference between them is $(1 - 0.01)$ m = 0.99 m.

(ii) 10^2 m = 100 m and 10^0 m = 1 m, so the difference between them is 99 m.

(iii) 10^4 m = 10 000 m and 10^2 m = 100 m, so the difference between them is 9900 m.

It is quite clear that as one goes up the scale the interval between each successive pair of tick marks increases by 100 times.

Question 2.5 (b)

The height of a child is about 10^0 m, i.e. 1 m. The height of Mount Everest is about 10^4 m (actually 8800 m, but it is not possible to read that accurately from the scale on Figure 2.2). So Mount Everest is $\sim 10^4$ times taller than a child.

Question 2.5 (c)

The length of a typical virus is 10^{-8} m and the thickness of a piece of paper is 10^{-4} m, so it would take $\sim 10^{-4}/10^{-8} = 10^{-4-(-8)} = 10^{-4+8} = 10^{4}$ viruses laid end to end to stretch across the thickness of a piece of paper.

Question 2.6

Magnitude 7 on the Richter scale represents four points more than magnitude 3, and each point increase represents a factor 10 increase in maximum ground movement. So a magnitude 7 earthquake corresponds to 10^4 (i.e. 10 000) times more ground movement than a magnitude 3 earthquake.

Question 2.7

Each of the quantities is quoted to four significant figures.

Question 2.8 (a)

The third digit is an 8, so the second digit must be rounded up:

$-38.87\ ^{\circ}\mathrm{C} = -39\ ^{\circ}\mathrm{C}$ to two significant figures

Question 2.8 (b)

There is no way of expressing a number greater than or equal to 100 unambiguously to two significant figures except by the use of scientific notation. The third digit is a 5, so again the second digit must be rounded up.

$$\begin{aligned} -195.8\ ^{\circ}\text{C} &= -1.958 \times 10^{2}\ ^{\circ}\text{C} \\ &= -2.0 \times 10^{2}\ ^{\circ}\text{C to two significant figures} \end{aligned}$$

{Note that the final zero does count.}

Question 2.8 (c)

Again, this quantity cannot be expressed unambiguously to two significant figures without the use of scientific notation. The third digit is an 8, so the second digit must be rounded up.

$$\begin{aligned}1083.4\ ^\circ\text{C} &= 1.0834 \times 10^3\ ^\circ\text{C}\\ &= 1.1 \times 10^3\ ^\circ\text{C to two significant figures}\end{aligned}$$

Question 3.1

(inch)^2, cm^2 and square miles all have units of (length)^2, so they are all units of area.

s^2 cannot be a unit of area because the unit which has been squared, the second, is a unit of time not of length.

m^{-2} cannot be a unit of area because the metre is raised to the power *minus* 2, not 2.

km^3 cannot be a unit of area because the kilometre is cubed not squared. In fact, it is a unit of volume.

Question 3.2 (a)

$\frac{6.732}{1.51} = 4.458 = 4.46$ to three significant figures.

{6.732 is known to four significant figures, and 1.51 is known to three significant figures. The number of significant figures in the answer is the same as in the input value with the fewest significant figures, i.e. three.}

Question 3.2 (b)

$2.0 \times 2.5 = 5.0$ to two significant figures.

{2.0 and 2.5 are both given to two significant figures, so the answer is given to two significant figures too.}

Question 3.2 (c)

Working to three significant figures and rounding to two significant figures at the end of the calculation gives:

$$\left(\frac{4.2}{3.1}\right)^2 = (1.35)^2 = 1.82 = 1.8 \text{ to two significant figures.}$$

{Squaring is repeated multiplication, so it is reasonable to quote the final answer to two significant figures. However, working to two significant figures throughout introduces a sizeable rounding error and gives a final answer of 2.0.}

Question 3.2 (d)

The total mass = 3×1.5 kg = 4.5 kg.

{Note that you have exactly 3 bags of flour, so it would not be correct to round the answer to one significant figure.}

Question 3.3 (a)

$$\begin{aligned}(3.0 \times 10^{6}) \times (7.0 \times 10^{-2}) &= (3.0 \times 7.0) \times 10^{6+(-2)} \\ &= 21 \times 10^{4} \\ &= 2.1 \times 10^{5}\end{aligned}$$

{Note that 21×10^{4} is a correct numerical answer to the multiplication, but it is not given in scientific notation.}

Question 3.3 (b)

$$\frac{8 \times 10^4}{4 \times 10^{-1}} = \frac{8}{4} \times 10^{4-(-1)} = 2 \times 10^5$$

Question 3.3 (c)

$$\frac{10^4 \times (4 \times 10^4)}{1 \times 10^{-5}} = 4 \times \frac{10^{4+4}}{10^{-5}} = 4 \times 10^{8-(-5)} = 4 \times 10^{13}$$

Question 3.3 (d)

$$
\begin{aligned}
\left(3.00 \times 10^{8}\right)^{2} &= (3.00)^{2} \times \left(10^{8}\right)^{2} \\
&= 9.00 \times 10^{8\times 2} \\
&= 9.00 \times 10^{16}
\end{aligned}
$$

Question 3.4

$$
\begin{aligned}
\text{Area} &= \left(9.78 \times 10^{-3}\ \text{m}\right)^2 \\
&= \left(9.78 \times 10^{-3}\right)^2 \text{m}^2 \\
&= 9.56 \times 10^{-5}\ \text{m}^2 \text{ to three significant figures.}
\end{aligned}
$$

Question 3.5

To one significant figure,

$$\text{distance to Proxima Centauri} \approx 4 \times 10^{16}\ \text{m}$$
$$\text{distance to the Sun} \approx 2 \times 10^{11}\ \text{m}$$

Thus,

$$\begin{aligned}\frac{\text{distance to Proxima Centauri}}{\text{distance to the Sun}} &\approx \frac{4 \times 10^{16}\ \text{m}}{2 \times 10^{11}\ \text{m}} \\ &\approx \frac{4}{2} \times \frac{10^{16}\ \text{m}}{10^{11}\ \text{m}} \\ &\approx 2 \times 10^{16-11} \\ &\approx 2 \times 10^{5}\end{aligned}$$

Thus Proxima Centauri is approximately 2×10^5 times further away than the Sun.

Question 3.6 (a)

1 m = 100 cm, so 1 $m^2 = 100^2\ cm^2$

Thus 1.04 $m^2 = 1.04 \times 100^2\ cm^2 = 1.04 \times 10^4\ cm^2$

Question 3.6 (b)

$1\ \text{m} = 10^6\ \mu\text{m}$, so $1\ \text{m}^2 = \left(10^6\right)^2\ \mu\text{m}^2$

Thus $1.04\ \text{m}^2 = 1.04 \times \left(10^6\right)^2\ \mu\text{m}^2 = 1.04 \times 10^{12}\ \mu\text{m}^2$

Question 3.6 (c)

$1\ \text{km} = 10^3\ \text{m}$, so $1\ \text{km}^2 = \left(10^3\right)^2\ \text{m}^2$

Thus $1\ \text{m}^2 = \dfrac{1}{(10^3)^2}\ \text{km}^2$

and $1.04\ \text{m}^2 = \dfrac{1.04}{(10^3)^2}\ \text{km}^2 = 1.04 \times 10^{-6}\ \text{km}^2$

Question 3.7 (a)

1 km = 10^3 m, so 1 $\text{km}^3 = \left(10^3\right)^3 \text{ m}^3 = 10^9 \text{ m}^3$

$$\begin{aligned}\text{Volume of Mars} &= 1.64 \times 10^{11} \text{ km}^3 \\ &= 1.64 \times 10^{11} \times 10^9 \text{ m}^3 \\ &= 1.64 \times 10^{20} \text{ m}^3\end{aligned}$$

Question 3.7 (b)

$1 \text{ m} = 10^3 \text{ mm, so } 1 \text{ m}^3 = \left(10^3\right)^3 \text{ mm}^3 = 10^9 \text{ mm}^3$

$$\text{Thus } 1 \text{ mm}^3 = \frac{1}{10^9} \text{ m}^3 = 10^{-9} \text{ m}^3$$

$$\begin{aligned}\text{Volume of ball bearing} &= 16 \text{ mm}^3 \\ &= 16 \times 10^{-9} \text{ m}^3 \\ &= 1.6 \times 10^{-8} \text{ m}^3\end{aligned}$$

Question 3.8 (a)

$$1 \text{ m} = 100 \text{ cm}$$

So

$$1 \text{ cm} = \frac{1}{100} \text{ m}$$

Thus

$$1 \text{ cm day}^{-1} = \frac{1}{100} \text{ m day}^{-1}$$

and

$$\begin{aligned} 12 \text{ cm day}^{-1} &= \frac{12}{100} \text{ m day}^{-1} \\ &= 0.12 \text{ m day}^{-1} \end{aligned}$$

Question 3.8 (b)

$$1 \text{ day} = 24 \times 60 \times 60 \text{ s} = 8.64 \times 10^4 \text{ s}$$

So

$$1 \text{ cm day}^{-1} = \frac{1}{8.64 \times 10^4} \text{ cm s}^{-1}$$

and

$$\begin{aligned} 12 \text{ cm day}^{-1} &= \frac{12}{8.64 \times 10^4} \text{ cm s}^{-1} \\ &= 1.4 \times 10^{-4} \text{ cm s}^{-1} \end{aligned}$$

Question 3.9 (a)

$$1\ \text{m} = 10^{3}\ \text{mm, so } 1\ \text{mm} = \frac{1}{10^{3}}\ \text{m} = 10^{-3}\ \text{m}$$

$$1\ \text{year} = 365 \times 24 \times 60 \times 60\ \text{s} = 3.154 \times 10^{7}\ \text{s}$$

To convert from mm year^{-1} m s^{-1} we need to *multiply* by 10^{-3} (to convert the mm to m) and *divide* by 3.154×10^{7} (to convert the year^{-1} to s^{-1}).

$$1\ \text{mm year}^{-1} = \frac{10^{-3}}{3.154 \times 10^{7}}\ \text{m s}^{-1}$$

so

$$\begin{aligned} 0.1\ \text{mm year}^{-1} &= 0.1 \times \frac{10^{-3}}{3.154 \times 10^{7}}\ \text{m s}^{-1} \\ &= 3 \times 10^{-12}\ \text{m s}^{-1} \text{ to one significant figure} \end{aligned}$$

So the stalactite is growing at about $3 \times 10^{-12}\ \text{m s}^{-1}$.

Question 3.9 (b)

$$1\ \text{m} = 100\ \text{cm, so } 1\ \text{cm} = \frac{1}{100}\ \text{m} = 10^{-2}\ \text{m}$$

$$1\ \text{day} = 24 \times 60 \times 60\ \text{s} = 8.64 \times 10^{4}\ \text{s}$$

To convert from cm day^{-1} to m s^{-1} we need to *multiply* by 10^{-2} (to convert the cm to m) and *divide* by 8.64×10^4 (to convert the day^{-1} to s^{-1}).

$$1\ \text{cm day}^{-1} = \frac{10^{-2}}{8.64 \times 10^{4}}\ \text{m s}^{-1}$$

$$\begin{aligned} 12\ \text{cm day}^{-1} &= 12 \times \frac{10^{-2}}{8.64 \times 10^{4}}\ \text{m s}^{-1} \\ &= 1.4 \times 10^{-6}\ \text{m s}^{-1} \end{aligned}$$

So the glacier is moving at about 1.4×10^{-6} m s^{-1}.

Question 3.9 (c)

$$1 \text{ km} = 10^3 \text{ m}$$

$$1 \text{ Ma} = 10^6 \times 365 \times 24 \times 60 \times 60 \text{ s} = 3.154 \times 10^{13} \text{ s}$$

To convert from km Ma^{-1} to m s^{-1}, we need to *multiply* by 10^3 (to convert the km to m) and *divide* by 3.154×10^{13} (to convert the Ma^{-1} to s^{-1}).

$$1 \text{ km Ma}^{-1} = \frac{10^3}{3.154 \times 10^{13}} \text{ m s}^{-1}$$

$$\begin{aligned} 35 \text{ km Ma}^{-1} &= 35 \times \frac{10^3}{3.154 \times 10^{13}} \text{ m s}^{-1} \\ &= 1.1 \times 10^{-9} \text{ m s}^{-1} \text{ to two significant figures.} \end{aligned}$$

So the plates are moving apart at an average rate of $1.1 \times 10^{-9} \text{ m s}^{-1}$.

Comparing the answers to parts (a), (b) and (c) shows that the tectonic plates are moving apart approximately 300 times faster than the stalactite is growing. The glacier under consideration moves about 1000 times faster still, but remember that there is considerable variation in the speeds at which all of these processes take place.

Question 3.10 (a)

$1\ \text{l} = 10^3\ \text{ml}$

To convert from $\mu\text{g}\,\text{l}^{-1}$ to $\mu\text{g}\,\text{ml}^{-1}$ we need to *divide* by 10^3.

$$1\ \mu\text{g}\,\text{l}^{-1} = \frac{1}{10^3}\ \mu\text{g}\,\text{ml}^{-1} = 10^{-3}\ \mu\text{g}\,\text{ml}^{-1}$$

$$\begin{aligned} 10\ \mu\text{g}\,\text{l}^{-1} &= 10 \times 10^{-3}\ \mu\text{g}\,\text{ml}^{-1} \\ &= 1.0 \times 10^{-2}\ \mu\text{g}\,\text{ml}^{-1} \text{ to two significant figures.} \end{aligned}$$

Question 3.10 (b)

Note that 10 $\mu g\,l^{-1}$ = 10 $\mu g\,dm^{-3}$, since 1 litre is defined to be equal to 1 dm^3 (Section 3.4.2).

$$1\ mg = 10^3\ \mu g$$

so

$$1\ \mu g = \frac{1}{10^3}\ mg = 10^{-3}\ mg$$

To convert from $\mu g\,dm^3$ to $mg\,dm^3$ we need to *multiply* by 10^{-3}.

$$1\ \mu g\,dm^3 = 10^{-3}\ mg\,dm^3$$

$$\begin{aligned}10\ \mu g\,dm^3 &= 10 \times 10^{-3}\ mg\,dm^3\\ &= 1.0 \times 10^{-2}\ mg\,dm^3 \text{ to two significant figures.}\end{aligned}$$

So a concentration of 10 $\mu g\,l^{-1}$ is equal to 1.0×10^{-2} $mg\,dm^3$.

Question 3.10 (c)

Note that 10 μg l^{-1} = 10 μg dm^{-3}.

$$1\ \text{g} = 10^6\ \mu\text{g}$$

$$\text{so } 1\ \mu\text{g} = \frac{1}{10^6}\ \text{g} = 10^{-6}\ \text{g}$$

$$1\ \text{m} = 10\ \text{dm}$$

$$\text{so } 1\ \text{m}^3 = 10^3\ \text{dm}^3$$

$$\text{and } 1\ \text{dm}^3 = \frac{1}{10^3}\ \text{m}^3 = 10^{-3}\ \text{m}^3$$

To convert from μg dm^{-3} to g m^{-3} we need to *multiply* by 10^{-6} (to convert the μg to g) and *divide* by 10^{-3} (to convert the dm^{-3} to m^{-3}).

$$1\ \mu\text{g}\,\text{dm}^{-3} = \frac{10^{-6}}{10^{-3}}\ \text{g}\,\text{m}^{-3}$$

$$
\begin{aligned}
10\ \mu\mathrm{g\,dm^{-3}} &= 10 \times \frac{10^{-6}}{10^{-3}}\ \mathrm{g\,m^{-3}} \\
&= 10 \times 10^{-6-(-3)}\ \mathrm{g\,m^{-3}} \\
&= 10 \times 10^{-3}\ \mathrm{g\,m^{-3}} \\
&= 1.0 \times 10^{-2}\ \mathrm{g\,m^{-3}} \text{ to two significant figures.}
\end{aligned}
$$

So a concentration of 10 $\mu\mathrm{g\,l^{-1}}$ is equal to $1.0 \times 10^{-2}\ \mathrm{g\,m^{-3}}$.

Question 3.11

(i) and (iii) are equivalent. Multiplication is commutative, so $x(y + z) = (y + z)x$

(ii) and (v) are equivalent. Both multiplication and addition are commutative, so $xy + z = z + yx$

Note that (i) is not equivalent to (ii) since, in (i), the whole of $(y + z)$, not just y, is multiplied by x.

Substituting $x = 3$, $y = 4$ and $z = 5$ gives

(i) $a = x(y + z) = 3 \times (4 + 5) = 27$

(ii) $a = xy + z = (3 \times 4) + 5 = 17$

(iii) $a = (y + z)x = (4 + 5) \times 3 = 27$

(iv) $a = x + yz = 3 + (4 \times 5) = 23$

(v) $a = z + yx = 5 + (4 \times 3) = 17$

Question 3.12

The equivalent equations are (i) and (iii), since

$$a\frac{bc^2}{d} = \frac{abc^2}{d} = \frac{bac^2}{d}$$

Note that only the c is squared, so (ii) $m = a\frac{b^2c^2}{d}$ and (v) $m = \frac{b^2a^2c^2}{d}$ are different.

Only the numerator of the fraction is multiplied by a, so (iv) $m = \frac{abc^2}{ad}$ is different too.

Question 3.13

$NPP = 1.06 \times 10^8$ kJ

$R = 3.23 \times 10^7$ kJ

From Equation 3.8,

$$GPP = NPP + R$$
$$= 1.06 \times 10^8 \text{ kJ} + 3.23 \times 10^7 \text{ kJ}$$
$$= 1.38 \times 10^8 \text{ kJ to three significant figures.}$$

Question 3.14

$\lambda = 621$ nm, $f = 4.83 \times 10^{14}$ Hz

Converting to SI base units gives

$$\lambda = 621 \times 10^{-9}\ \text{m} = 6.21 \times 10^{-7}\ \text{m}$$
$$f = 4.83 \times 10^{14}\ \text{Hz} = 4.83 \times 10^{14}\ \text{s}^{-1}$$

From Equation 3.13,

$$\begin{aligned} v &= f\lambda \\ &= 4.83 \times 10^{14}\ \text{s}^{-1} \times 6.21 \times 10^{-7}\ \text{m} \\ &= 3.00 \times 10^{8}\ \text{m s}^{-1} \text{ to three significant figures.} \end{aligned}$$

{Note that this is the speed of light in a vacuum. Light of this frequency and wavelength is in the red part of the visible spectrum.}

Question 3.15 (a)

From Equation 3.5

$$V = \frac{4}{3}\pi r^3$$
$$r = 6.38 \times 10^3 \text{ km} = 6.38 \times 10^3 \times 10^3 \text{ m} = 6.38 \times 10^6 \text{ m}$$

So

$$V = \frac{4}{3}\pi \left(6.38 \times 10^6 \text{ m}\right)^3$$
$$= 1.09 \times 10^{21} \text{ m}^3 \text{ to three significant figures.}$$

The Earth's volume is 1.09×10^{21} m^3.

Question 3.15 (b)

From Equation 3.18

$$F_g = G\,\frac{m_1 m_2}{r^2}$$

$$G = 6.673 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$$

$$m_1 = 5.97 \times 10^{24}\ \mathrm{kg}$$

$$m_2 = 7.35 \times 10^{22}\ \mathrm{kg}$$

$$\begin{aligned} r &= 3.84 \times 10^{5}\ \mathrm{km} \\ &= 3.84 \times 10^{5} \times 10^{3}\ \mathrm{m} \\ &= 3.84 \times 10^{8}\ \mathrm{m} \end{aligned}$$

Substituting values into the equation gives

$$F_g = 6.673 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}} \times \frac{5.97 \times 10^{24}\ \mathrm{kg} \times 7.35 \times 10^{22}\ \mathrm{kg}}{(3.84 \times 10^{8}\ \mathrm{m})^2}$$

Rearranging to collect the units together

$$F_g = \frac{6.673 \times 10^{-11} \times 5.97 \times 10^{24} \times 7.35 \times 10^{22}\ \mathrm{N\,m^2\,kg^{-2}\,kg\,kg}}{(3.84 \times 10^{8})^2\ \mathrm{m^2}}$$

Many of the units can be cancelled

$$F_g = \frac{6.673 \times 10^{-11} \times 5.97 \times 10^{24} \times 7.35 \times 10^{22}\ \mathrm{N}\,\cancel{\mathrm{m}^2}\,\cancel{\mathrm{kg}^{-2}}\,\cancel{\mathrm{kg}}\,\cancel{\mathrm{kg}}}{(3.84 \times 10^8)^2\ \cancel{\mathrm{m}^2}}$$

Calculating the numeric value gives

$$F_g = 1.99 \times 10^{20}\ \mathrm{N} \text{ to 3 significant figures.}$$

{Note that there was no need to express the newtons in terms of base units on this occasion; all the other units cancelled to leave N as the units of force, as expected.}

The magnitude of the gravitational force between the Earth and the Moon is 1.99×10^{20} N.

Question 4.1 (a)

$v = f\lambda$ can be reversed to give $f\lambda = v$.

To isolate f we need to remove λ, and f is currently *multiplied* by λ so, according to Hint 3, we need to *divide* by λ. Remember that we must do this to *both sides of the equation*, so we have

$$\frac{f\lambda}{\lambda} = \frac{v}{\lambda}$$

The λ in the numerator of the fraction on the left-hand side cancels with the λ in the denominator to give

$$f = \frac{v}{\lambda}$$

Question 4.1 (b)

E_{tot} = can be reversed to give $E_k + E_p = E_{tot}$.

To isolate E_k we need to remove E_p, and E_p is currently *added* to E_k so, according to Hint 1, we need to *subtract* E_p. Remember that we must do this to *both sides of the equation*, so we have

$$E_k + E_p - E_p = E_{tot} - E_p$$

$E_p - E_p = 0$, so

$$E_k = E_{tot} - E_p$$

Question 4.1 (c)

$\rho = \frac{m}{V}$ can be reversed to give $\frac{m}{V} = \rho$

To isolate m we need to remove V, and m is currently *divided* by V so, according to Hint 4, we need to *multiply* by V. Remember that we must do this to *both sides of the equation*, so we have

$$\frac{mV}{V} = \rho V$$

The V in the numerator of the fraction on the left-hand side cancels with the V in the denominator to give

$$m = \rho V$$

Question 4.2 (a)

$b = c - d + e$ can be written as $c - d + e = b$ (with e on the left-hand side).

Adding d to both sides gives

$$c - d + e + d = b + d$$

i.e.

$$c + e = b + d$$

Subtracting c from both sides gives

$$c + e - c = b + d - c$$

i.e.

$$e = b + d - c.$$

Question 4.2 (b)

$p = \rho g h$ can be written as $\rho g h = p$ (with h on the left-hand side).

Dividing both sides by ρ gives

$$\frac{\rho g h}{\rho} = \frac{p}{\rho}$$

i.e.

$$g h = \frac{p}{\rho}$$

Dividing both sides by g gives

$$\frac{g h}{g} = \frac{p}{\rho g}$$

i.e.

$$h = \frac{p}{\rho g}$$

Question 4.2 (c)

$$v_{\text{esc}}^2 = \frac{2GM}{R}$$

Multiplying both sides by R (to get R onto the left-hand side) gives

$$\begin{aligned} v_{\text{esc}}^2 R &= \frac{2GMR}{R} \\ &= 2GM \end{aligned}$$

Dividing both sides by v_{esc}^2 gives

$$\frac{v_{\text{esc}}^2 R}{v_{\text{esc}}^2} = \frac{2GM}{v_{\text{esc}}^2}$$

i.e.

$$R = \frac{2GM}{v_{\text{esc}}^2}$$

Question 4.2 (d)

$$E = hf - \phi$$

Adding ϕ to both sides (to get ϕ onto the left-hand side) gives

$$E + \phi = hf - \phi + \phi$$

i.e.

$$E + \phi = hf$$

Subtracting E from both sides gives

$$E + \phi - E = hf - E$$

that is

$$\phi = hf - E$$

Question 4.2 (e)

We need to start by finding an equation for c^2.

$a = \dfrac{bc^2}{d}$ can be written as $\dfrac{bc^2}{d} = a$ (with c on the left-hand side).

Multiplying both sides by d gives

$$\frac{bc^2 d}{d} = ad$$

i.e.

$$bc^2 = ad$$

Dividing both sides by b gives

$$\frac{bc^2}{b} = \frac{ad}{b}$$

i.e.

$$c^2 = \frac{ad}{b}$$

Taking the square root of both sides gives

$$c = \pm\sqrt{\frac{ad}{b}}$$

Question 4.2 (f)

$a = \sqrt{\frac{b}{c}}$ can be written as $\sqrt{\frac{b}{c}} = a$ (with b on the left-hand side)

Squaring both sides gives

$$\frac{b}{c} = a^2$$

Multiplying both sides by c gives

$$\frac{bc}{c} = a^2c$$

i.e.

$$b = a^2c$$

Question 4.3 (a)

We need to start by finding an equation for v^2.

$E_\mathrm{k} = \frac{1}{2}mv^2$ can be written as $\frac{1}{2}mv^2 = E_\mathrm{k}$. (with the v^2 on the left-hand side).

Multiplying both sides by 2 gives

$$mv^2 = 2E_\mathrm{k}$$

Dividing both sides by m gives

$$v^2 = \frac{2E_\mathrm{k}}{m}$$

Taking the square root of both sides gives

$$v = \pm\sqrt{\frac{2E_\mathrm{k}}{m}}$$

but we are only interested in the positive value on this occasion.

Question 4.3 (b)

If $E_k = 2 \times 10^3$ J and $m = 4 \times 10^{21}$ kg

$$
\begin{aligned}
v &= \sqrt{\frac{2E_k}{m}} \\
&= \sqrt{\frac{2 \times 2 \times 10^3 \text{ J}}{4 \times 10^{21} \text{ kg}}} \\
&= \sqrt{1 \times 10^{-18} \frac{\cancel{\text{kg}}\,\text{m}^2\,\text{s}^{-2}}{\cancel{\text{kg}}}} \\
&= 1 \times 10^{-9} \text{ m s}^{-1}
\end{aligned}
$$

{At this speed, the plate would move 3 cm in a year.}

Question 4.3 (c)

If $E_\mathrm{k} = 2 \times 10^3$ J and $m = 70$ kg

$$v = \sqrt{\frac{2E_\mathrm{k}}{m}}$$

$$= \sqrt{\frac{2 \times 2 \times 10^3\ \mathrm{J}}{70\ \mathrm{kg}}}$$

$$= 8\ \mathrm{m\,s^{-1}}$$

{The sprinter, having a smaller mass, has to move rather faster than the tectonic plate!}

Question 4.4 (a)

$v_x = u_x + a_x t$ can be written as

$$u_x + a_x t = v_x$$

Subtracting u_x from both sides gives

$$a_x t = v_x - u_x$$

Dividing both sides by t gives

$$a_x = \frac{v_x - u_x}{t}$$

Question 4.4 (b)

Squaring both sides of $v_\mathrm{s} = \sqrt{\frac{\mu}{\rho}}$ gives

$$v_s^2 = \frac{\mu}{\rho}$$

Multiplying both sides by ρ gives

$$\rho\, v_s^2 = \mu$$

Dividing both sides by v_s^2 gives

$$\rho = \frac{\mu}{v_s^2}$$

Question 4.4 (c)

Multiplying both sides of $F = \dfrac{L}{4\pi d^2}$ by d^2 gives

$$Fd^2 = \frac{L}{4\pi}$$

Dividing both sides by F gives

$$d^2 = \frac{L}{4\pi F}$$

Taking the square root of both sides gives

$$d = \pm\sqrt{\frac{L}{4\pi F}}$$

{Note that if we consider just the positive value, we have arrived at Equation 3.20, albeit written rather differently.}

Question 4.5 (a)

$$\frac{\mu_0}{2\pi} \times \frac{i_1 i_2}{d} = \frac{\mu_0 \times i_1 i_2}{2\pi \times d} = \frac{\mu_0 i_1 i_2}{2\pi d}$$

Question 4.5 (b)

Note that $\dfrac{3a}{2b}\bigg/2$ means $\dfrac{3a}{2b}$ divided by 2.

$$\frac{3a}{2b}\bigg/2 = \frac{3a}{2b} \times \frac{1}{2} = \frac{3a}{4b}$$

Question 4.5 (c)

The product $c \times b$ will be a common denominator, so we can write

$$\frac{2b}{c} + \frac{3c}{b} = \frac{2b \times b}{c \times b} + \frac{3c \times c}{b \times c} = \frac{2b^2 + 3c^2}{cb}$$

This is the simplest form in which this fraction can be expressed.

Question 4.5 (d)

$$\frac{2ab}{c} \div \frac{2ac}{b} = \frac{2ab}{c} \times \frac{b}{2ac}$$

Cancelling the '2a's gives

$$\frac{2ab}{c} \div \frac{2ac}{b} = \frac{\cancel{2a}b}{c} \times \frac{b}{\cancel{2a}c} = \frac{b^2}{c^2}$$

{Note that, for all parts of Question 4.5 and for many other questions involving simplification, it is possible to check that the algebraic expression you end up with is equivalent to the one that you started with by substituting numerical values for the variables. For example, setting $a = 2$, $b = 3$ and $c = 4$ in the original expression gives

$$\begin{aligned}\frac{2ab}{c} \div \frac{2ac}{b} &= \left(\frac{2 \times 2 \times 3}{4}\right) \div \left(\frac{2 \times 2 \times 4}{3}\right) \\ &= \frac{12}{4} \div \frac{16}{3} = 3 \div \frac{16}{3} = 3 \times \frac{3}{16} = \frac{9}{16}\end{aligned}$$

Substituting the same values in the answer gives $\dfrac{b^2}{c^2} = \dfrac{3^2}{4^2} = \dfrac{9}{16}$ }

Question 4.5 (e)

The product $f(f+1)$ will be a common denominator, so we can write

$$\begin{aligned}\frac{1}{f}-\frac{1}{f+1} &= \frac{(f+1)}{f(f+1)}-\frac{f}{(f+1)f}\\ &= \frac{f+1-f}{f(f+1)}\\ &= \frac{1}{f(f+1)}\end{aligned}$$

Question 4.5 (f)

$$\frac{2b^2}{(b+c)} \div \frac{2c^2}{(a+c)} = \frac{\cancel{2}b^2}{(b+c)} \times \frac{(a+c)}{\cancel{2}c^2}$$
$$= \frac{b^2(a+c)}{c^2(b+c)}$$

The expression can be written as $\left(\frac{b}{c}\right)^2 \frac{(a+c)}{(b+c)}$ but cannot be simplified further.

Question 4.6

The equation can be written as

$$\begin{aligned}\frac{1}{f} &= \frac{1}{u} + \frac{1}{v}\\ &= \frac{v}{uv} + \frac{u}{vu} \quad \text{(taking the product } uv \text{ as the common denominator)}\\ &= \frac{v+u}{uv}\end{aligned}$$

Taking the reciprocal of both sides of the equation gives

$$f = \frac{uv}{v+u}$$

Question 4.7 (a)

$$\frac{1}{2}(v_x + u_x)\,t = \frac{1}{2}v_x t + \frac{1}{2}u_x t$$

or alternatively

$$\frac{1}{2}(v_x + u_x)\,t = \frac{v_x t}{2} + \frac{u_x t}{2} \text{ or } \frac{v_x t + u_x t}{2}$$

Question 4.7 (b)

$$\frac{(a-b)-(a-c)}{2} = \frac{a-b-a+c}{2}$$
$$= \frac{c-b}{2}$$

since $a - a = 0$, and $-b + c$ is more tidily written as $c - b$.

Question 4.7 (c)

$$\begin{aligned}(k-2)(k-3) &= k^2 - 3k - 2k + 6 \\ &= k^2 - 5k + 6\end{aligned}$$

Question 4.7 (d)

$$
\begin{aligned}
(t-2)^2 &= (t-2)(t-2) \\
&= t^2 - 2t - 2t + 4 \\
&= t^2 - 4t + 4
\end{aligned}
$$

Question 4.8 (a)

$$y^2 - y = y(y - 1)$$

Question 4.8 (b)

$x^2 - 25 = (x + 5)(x - 5)$, by comparison with Equation 4.3.

We can check that the factorization is correct by multiplying the brackets out. This gives

$$
\begin{aligned}
(x+5)(x-5) &= x^2 - 5x + 5x - 25 \\
&= x^2 - 25
\end{aligned}
$$

Question 4.9

Both the terms on the right-hand side of $E_{\text{tot}} = \frac{1}{2}mv^2 + mg\,\Delta h$ include m, so we can rewrite the equation as

$$E_{\text{tot}} = m\left(\frac{1}{2}v^2 + g\,\Delta h\right)$$

Reversing the order gives

$$m\left(\frac{1}{2}v^2 + g\,\Delta h\right) = E_{\text{tot}}$$

Dividing both sides by $\left(\frac{1}{2}v^2 + g\,\Delta h\right)$ gives

$$m = \frac{E_{tot}}{\frac{1}{2}v^2 + g\,\Delta h}$$

This is a perfectly acceptable equation for m, but the fraction in the denominator looks a little untidy. Multiplying the numerator and denominator by 2 gives

$$m = \frac{2E_{tot}}{v^2 + 2g\,\Delta h}$$

Question 4.10 (a)

From the answer to Question 4.7 (c)

$k^2 - 5k + 6 = (k - 2)(k - 3)$

Thus, if $k^2 - 5k + 6 = 0$, then $(k - 2)(k - 3) = 0$ too,
so $k - 2 = 0$ or $k - 3 = 0$.
i.e. $k = 2$ or $k = 3$

Checking for $k = 2$:
$k^2 - 5k + 6 = 2^2 - (5 \times 2) + 6 = 4 - 10 + 6 = 0$, as expected.
Checking for $k = 3$:
$k^2 - 5k + 6 = 3^2 - (5 \times 3) + 6 = 9 - 15 + 6 = 0$, as expected.

So the solutions of the equation $k^2 - 5k + 6 = 0$ are $k = 2$ and $k = 3$.

Question 4.10 (b)

From the answer to Question 4.7 (d)

$t^2 - 4t + 4 = (t - 2)^2$

Thus, if $t^2 - 4t + 4 = 0$, then $(t - 2)^2 = 0$ too,
so $t - 2 = 0$,
i.e. $t = 2$.

Checking:
$t = 2$ gives $t^2 - 4t + 4 = 2^2 - (4 \times 2) + 4 = 4 - 8 + 4 = 0$, as expected.

So the solution of the equation $t^2 - 4t + 4 = 0$ is $t = 2$.

Question 4.10 (c)

Comparison of $k^2 - 5k + 6 = 0$ with $ax^2 + bx + c = 0$ shows that $a = 1$, $b = -5$ and $c = 6$ on this occasion, so the solutions are

$$\begin{aligned} k &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-5) \pm \sqrt{(-5)^2 - (4 \times 1 \times 6)}}{2 \times 1} \\ &= \frac{5 \pm \sqrt{25 - 24}}{2} \\ &= \frac{5 \pm 1}{2} \end{aligned}$$

so $k = \dfrac{5+1}{2} = \dfrac{6}{2} = 3$ or $k = \dfrac{5-1}{2} = \dfrac{4}{2} = 2$.

So the solutions of the equation $k^2 - 5k + 6 = 0$ are $k = 2$ and $k = 3$. This is the same answer as was obtained in part (a) and could be checked in the same way.

Question 4.10 (d)

Comparison of $t^2 - 4t + 4 = 0$ with $ax^2 + bx + c = 0$ shows that $a = 1$, $b = -4$ and $c = 4$ on this occasion, so the solutions are

$$
\begin{aligned}
k &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-4) \pm \sqrt{(-4)^2 - (4 \times 1 \times 4)}}{2 \times 1} \\
&= \frac{4 \pm \sqrt{16 - 16}}{2} \\
&= \frac{4 \pm 0}{2} \\
&= 2
\end{aligned}
$$

So there is just one solution to $t^2 - 4t + 4 = 0$; namely $t = 2$. This is the same answer as was obtained in part (b) and could be checked in the same way.

Question 4.11 (a)

Rearranging $p = mv$ to make v the subject gives

$$v = \frac{p}{m} \quad \text{(dividing both sides by } m\text{)}$$

Substituting in $E_\text{k} = \frac{1}{2}mv^2$ gives

$$\begin{aligned} E_\text{k} &= \tfrac{1}{2}m\left(\frac{p}{m}\right)^2 \\ &= \tfrac{1}{2}m\frac{p^2}{m^2} \\ &= \frac{p^2}{2m} \end{aligned}$$

Question 4.11 (b)

Since both equations are already written with E (the variable we are trying to eliminate) as the subject, we can simply set the two equations for E equal to each other:

$$\tfrac{1}{2}mv^2 = mg\,\Delta h$$

There is an m on both sides of the equation; dividing both sides of the equation by m gives

$$\tfrac{1}{2}v^2 = g\,\Delta h$$

Multiplying both sides of the equation by 2 gives

$$v^2 = 2g\,\Delta h$$

Taking the square root of both sides of the equation gives

$$v = \pm\sqrt{2g\,\Delta h}$$

Question 4.11 (c)

Rearranging $c = f\lambda$ to make f the subject gives

$$f = \frac{c}{\lambda} \quad \text{(dividing both sides by } \lambda\text{)}$$

Substituting in $E_\mathrm{k} = hf - \phi$ gives

$$E_\mathrm{k} = \frac{hc}{\lambda} - \phi$$

Adding ϕ to both sides of the equation gives

$$E_\mathrm{k} + \phi = \frac{hc}{\lambda}$$

Subtracting E_k from both sides gives

$$\phi = \frac{hc}{\lambda} - E_\mathrm{k}$$

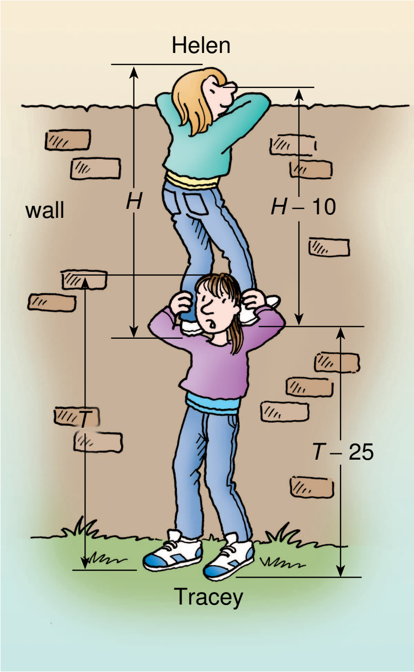
Question 4.12

Let the number selected be represented by x:

Adding 5 gives	$x + 5$
Doubling the result gives	$2(x + 5) = 2x + 10$
Subtracting 2 gives	$(2x + 10) - 2 = 2x + 8$
Dividing by 2 gives	$\frac{2x + 8}{2} = x + 4$

Taking away the number you first thought of gives $(x + 4) - x = 4$.

Question 4.13



Let H represent Helen's height in cm and T represent Tracey's height in cm. Since Tracey is 15 cm taller than Helen we can write

$$T = H + 15 \qquad \text{(i)}$$

The height of the wall is equal to Tracey's height up to her shoulders $(T - 25)$ plus Helen's height up to her eyes $(H - 10)$, thus

$$(T - 25) + (H - 10) = 300 \qquad \text{(ii)}$$

Simplifying (ii) gives

$$T + H - 35 = 300$$

Adding 35 to both sides gives

$$T + H = 335$$

Substituting for T from (i) gives

$$(H + 15) + H = 335$$
$$2H + 15 = 335$$

Subtracting 15 from both sides gives

$$2H = 320$$

Dividing both sides by 2 gives

$$H = 160$$

i.e. Helen is 160 cm tall.

Question 4.14

The equations required are $E_\mathrm{g} = mg\,\Delta h$ (Equation 4.18) and $E_\mathrm{k} = \frac{1}{2}mv^2$ (Equation 4.17).

Assuming that the child's gravitational potential energy is converted into kinetic energy, $E_\mathrm{k} = E_\mathrm{g}$.

$$\tfrac{1}{2}mv^2 = mg\,\Delta h$$

Dividing both sides by m gives

$$\tfrac{1}{2}v^2 = g\,\Delta h$$

Multiplying both sides by 2 gives

$$v^2 = 2g\,\Delta h$$

Taking the square root of both sides gives

$$v = \pm\sqrt{2g\,\Delta h}$$

On this occasion we are only interested in the positive square root, i.e. $v = \sqrt{2g\,\Delta h}$

Substituting $\Delta h = 1.8$ m and $g = 9.81$ m s^{-2} gives

$$\begin{aligned} v &= \sqrt{2 \times 9.81\ \mathrm{m\,s^{-2}} \times 1.8\ \mathrm{m}} \\ &= 5.9\ \mathrm{m\,s^{-1}}\ \text{to two significant figures} \end{aligned}$$

(noting that $\sqrt{\mathrm{m^2\,s^{-1}}} = \mathrm{m\,s^{-1}}$).

Checking

The units have worked out to be m s^{-1}, as expected.

An estimated value is

$$\begin{aligned} v &\approx \sqrt{2 \times 10\ \mathrm{m\,s^{-2}} \times 2\ \mathrm{m}} \\ &\approx \sqrt{40\ \mathrm{m^2\,s^{-2}}} \\ &\approx 6\ \mathrm{m\,s^{-1}},\ \text{since}\ \sqrt{40} \approx \sqrt{36} \end{aligned}$$

The speed seems quite high; in reality not all of the child's gravitational potential energy would be converted into kinetic energy.

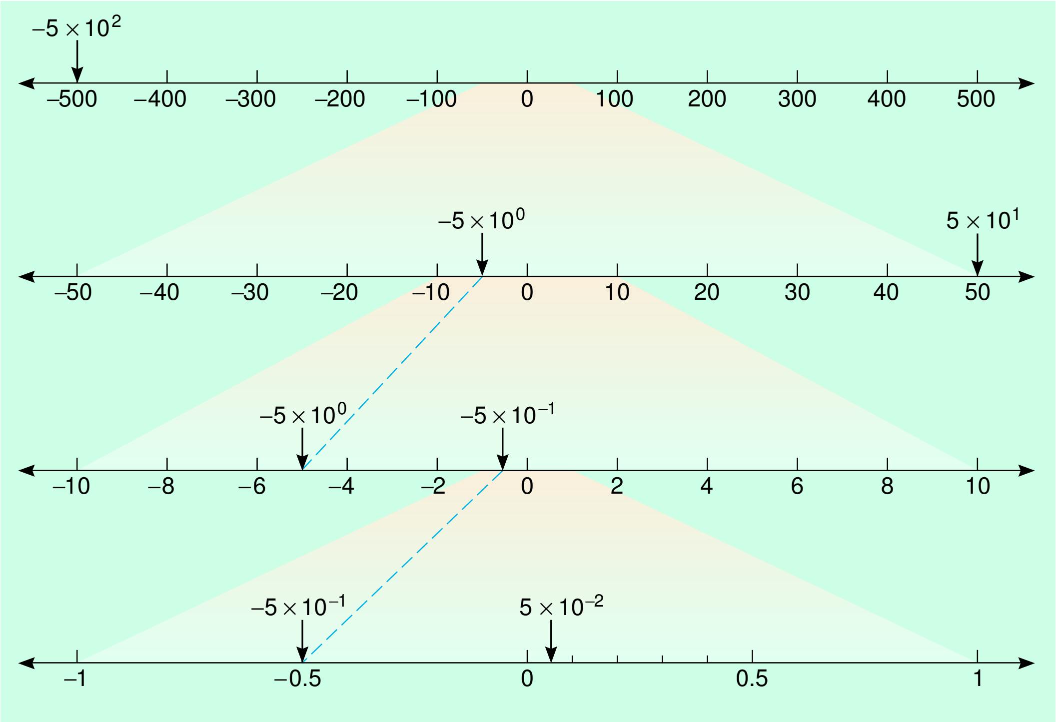


Figure 2.1: Portions of the number line, showing the positions of a few large and small numbers expressed in scientific notation.

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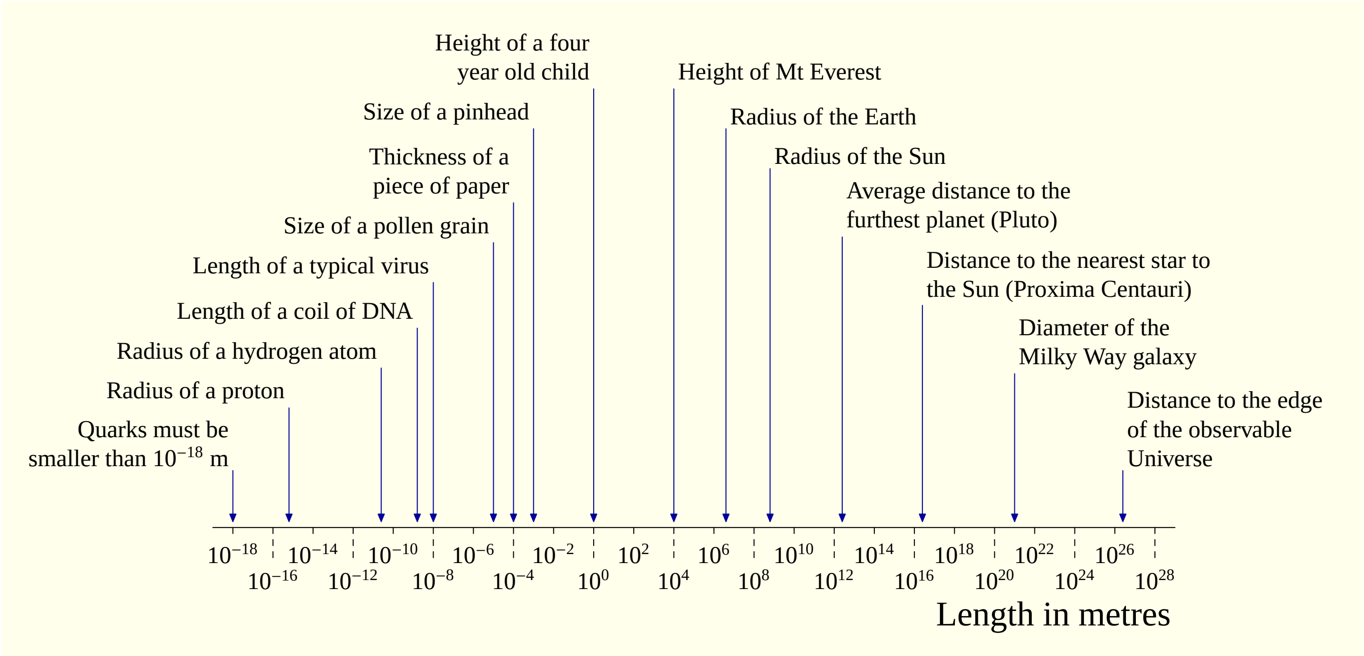

Figure 2.2: The scale of the known Universe.

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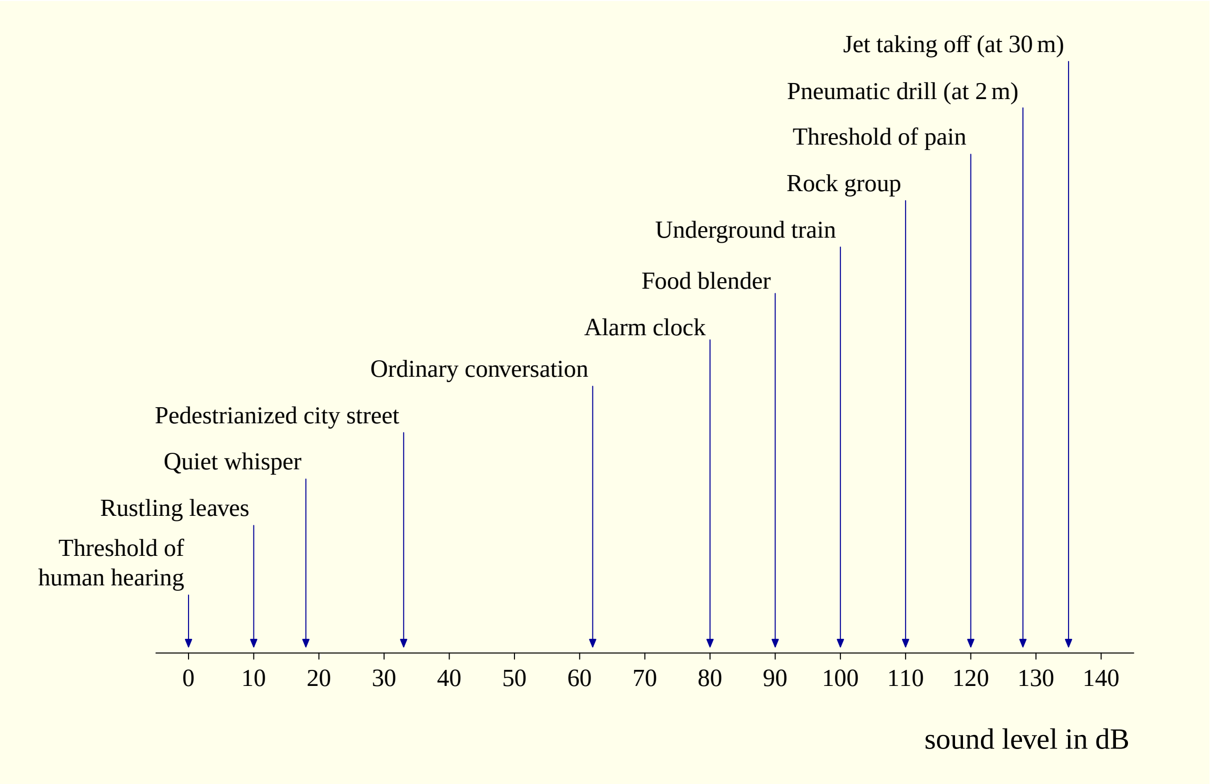

Figure 2.3: Some common sounds on the decibel scale of sound level.

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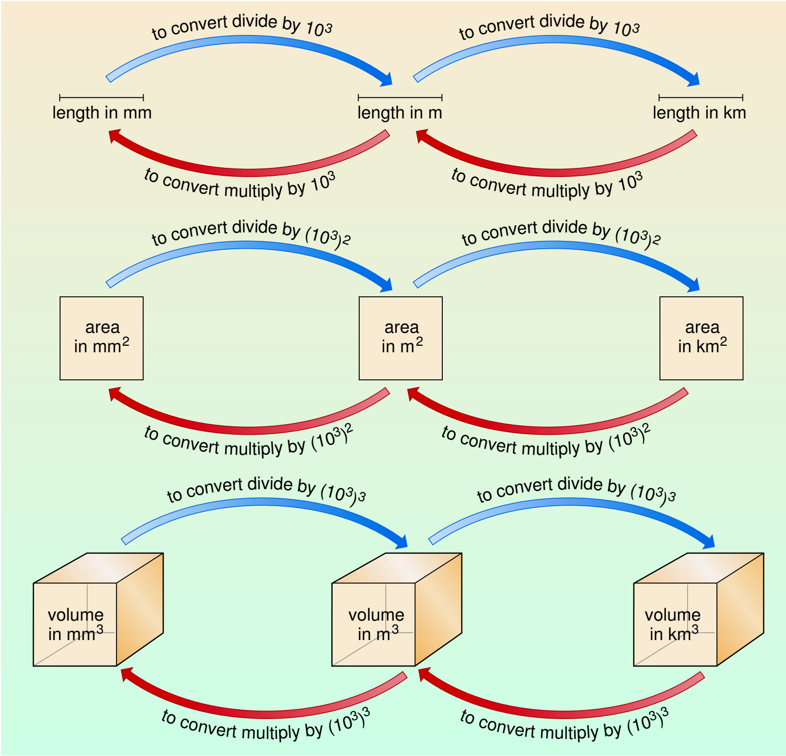

Figure 3.8: Unit conversions for length, area and volume.

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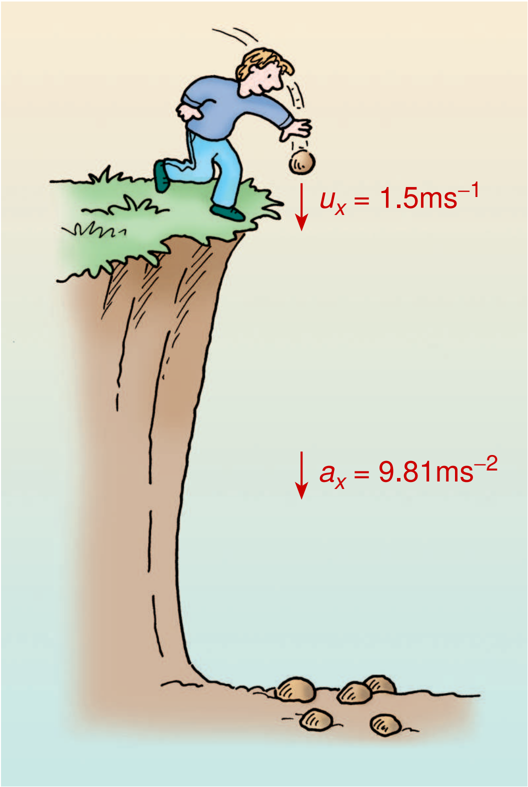

Figure 3.11: A stone being thrown from a cliff.

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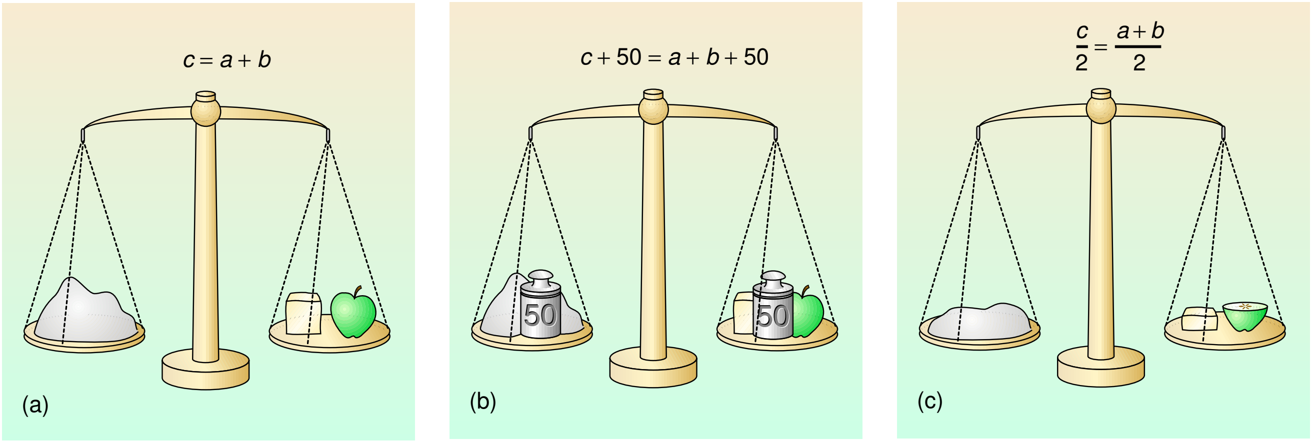

Figure 4.1: (a) The analogy between an equation and a set of kitchen scales. The scales remain balanced if (b) 50 g is added to both sides or if (c) the weight on both sides is halved.

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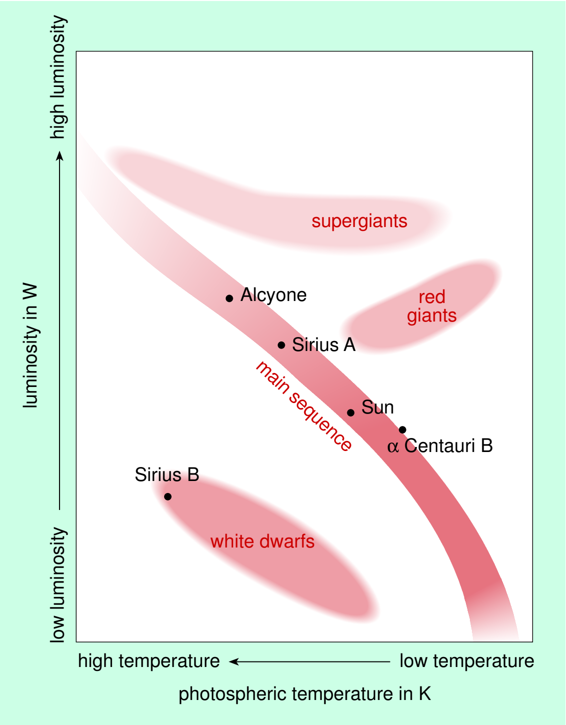

Figure 4.2: A Hertzsprung–Russell diagram showing the Sun and a number of other stars.

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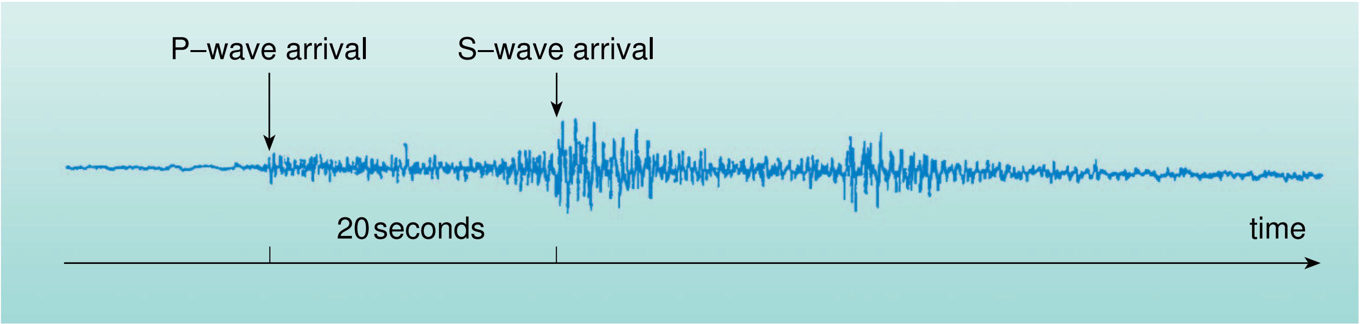

Figure 4.4: Seismogram recorded at the British Geological Survey in Edinburgh on 12 September 1988 at 2.23 p.m.

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Box 3.4 Some scientific formulae

$$C = 2\pi r \tag{3.3}$$

where C is the circumference of a circle of radius r.

$$A = \pi r^2 \tag{3.4}$$

where A is the area of a circle of radius r.

$$V = \frac{4}{3}\pi r^3 \tag{3.5}$$

where V is the volume of a sphere of radius r.

$$F = ma \tag{3.6}$$

where F is the magnitude of force on an object, m is its mass and a is the magnitude of its acceleration.

$$E = mc^2 \tag{3.7}$$

where E is energy, m is mass and c is the speed of light.

$$GPP = NPP + R \tag{3.8}$$

where GPP is the gross primary production of energy by plants in an ecosystem, NPP is net primary production and R is energy used in plant respiration.

$$\rho = \frac{m}{V} \tag{3.9}$$

where ρ is the density of an object of mass m and volume V.

$$v_{\mathrm{s}} = \sqrt{\frac{\mu}{\rho}} \tag{3.10}$$

where v_{s} is the speed of an S wave travelling through rocks of density ρ and rigidity modulus μ.

$$P = \rho g h \tag{3.11}$$

where P is the pressure at depth h in a liquid of density ρ, and g is the acceleration due to gravity.

$$PV = nRT \tag{3.12}$$

where P is the pressure of n moles of a gas in a container of volume V held at temperature T and R is a constant called the gas constant.

$$v = f\lambda \tag{3.13}$$

where v is the speed of a wave, f is its frequency and λ is its wavelength.

$$q = mc\,\Delta T \tag{3.14}$$

where q is the heat transferred to an object, m is its mass, c is its specific heat capacity and ΔT is the change in its temperature.

$$v_{av} = \frac{v_i + v_f}{2} \tag{3.15}$$

where v_{av} is average speed, v_i is initial speed and v_f is final speed.

$$v_x = u_x + a_x t \tag{3.16}$$

where u_x, v_x and a_x are respectively initial speed, final speed and acceleration, all in the direction of the x-axis, and t is time.

$$s_x = u_x t + \frac{1}{2} a_x t^2 \tag{3.17}$$

where s_x, u_x and a_x are respectively distance, initial speed and acceleration, all in the direction of the x-axis, and t is time.

$$F_g = G \frac{m_1 m_2}{r^2} \tag{3.18}$$

where F_g is the magnitude of the gravitational force between two objects of masses m_1 and m_2, a distance r apart. G is a constant called Newton's universal gravitational constant.

$$v_{\text{esc}} = \left(\frac{2GM}{R}\right)^{1/2} \tag{3.19}$$

where v_{esc} is the escape speed, i.e. the speed with which an object must be fired from the surface of a planet of mass M and radius R in order just to escape from it. G is Newton's universal gravitational constant.

$$d = [L/(4\pi F)]^{1/2} \tag{3.20}$$

where d is the distance at which light from a star of luminosity L has a flux density of F.

Return to Section 3.5.2

alpha	Α	α	nu (new)	Ν	ν
beta	Β	β	xi (csi)	Ξ	ξ
gamma	Γ	γ	omicron	Ο	ο
delta	Δ	δ	pi (pie)	Π	π
epsilon	Ε	ϵ	rho (roe)	Ρ	ρ
zeta	Ζ	ζ	sigma	Σ	σ
eta	Η	η	tau (taw)	Τ	τ
theta	Θ	θ	upsilon	Υ	υ
iota	Ι	ι	phi (fie)	Φ	ϕ
kappa	Κ	κ	chi (kie)	Χ	χ
lambda	Λ	λ	psi	Ψ	ψ
mu (mew)	Μ	μ	omega	Ω	ω

Table 3.1: The Greek alphabet. The pronunciation is given in parentheses where it is not obvious.

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